Survey on Some Singularly Perturbed Models Arising from Physical Problems

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Abstract : Singular perturbation problems arise as mathematical models in the various field of applied mathematics and engineering. In this work, some popular models of singular perturbation problems occurring in physical world have been described briefly.

IndexTerms **- Singular perturbation problems, ordinary differential equations, partial differential equations, boundary conditions.**

I. INTRODUCTION

In the field of chemistry, biology, physics and engineering many phenomenon can be defined by boundary value problems associated with different ordinary and partial differential equations. Singularly perturbed problems played an important role in the theory of differential equations and in their applications to the physical world. In 1904, Prandtl introduced the concept of singular perturbations at the Third International Congress of Mathematicians in Heidelberg [27]. However, the term singular perturbations was first used in the work of Friedrichs and Wasow [12]. Singularly perturbation problems occur in various branches of engineering and applied mathematics such as boundary layers in fluid dynamics, skin layers in electrical networks, gas porous electrodes theory, shock layers in fluid and solid mechanics, edge layers in solid mechanics, diffraction theory, transition points in quantum mechanics, aerodynamics, reaction-diffusion processes, elasticity, oceanography, chemical reactor-theory, plasma dynamics. In such problems, perturbation parameters plays a role in a narrow region where the solution changes rapidly. This give rise to the occurrence of typical initial, interior and boundary layers in the solution of singular perturbation problems and prevent us from obtaining the satisfactory numerical solutions. Asymptotic analysis and Numerical analysis and the two main approaches to obtain the solution of singularly perturbed problems. Asymptotic analysis provide us the qualitative behavior of a family of problems and only some quantitative information about any particular member of the family. On the other hand, Numerical analysis approach provides the quantitative information about a particular problem. Numerical methods dealt with a broad class of perturbation problems and also minimize demands upon the problem solver. Asymptotic methods treat comparatively restricted classes of problems and require the problem solver to have some understanding of the behavior of the solution expected.

In this paper, some standard singularly perturbed models arising from the physical world are presented.

II. SOME SINGULARLY PERTURBED MODELS

1. Black Scholes Equation :

The financial data can be modeled by using the Black-Scholes equation

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rS,\quad (S,t)\in\mathbf{R}^+\times[0,T),$$

with the boundary and final conditions

$$C(0,t)=0,\quad C(S,t)\to S,\quad \mathrm{t}\in[0,\mathrm{T}),$$
$$C(S,T)=\max(S-E,\ 0),\quad S\in\mathbf{R}^+,$$

where $C=C(S,t)$ denotes the European call option. S and t represents the current values of the asset and time respectively. $E,\ T,\ \sigma$ and r are used to denote the exercise price, expiry time, volatility and the risk free interest rate respectively [21].

2. Britton Model for Population Dynamics :

The singularly perturbed dynamical differential equation from population dynamics is

$$\frac{\partial v_\mu}{\partial t}=\mu\square v_\mu(1-f*v_\mu),$$

where

$$f * v_\mu = \int_{t-\sigma}^{t}\int_{\Omega} f(x-y,\ t-s)v_\mu(y,s)dyds,$$

where $v_\mu(x,t)$ is a population density that evolves through the reproduction and the random migration. The convolution operator is involved with a kernel $f(x,t)$ that models the distributed age structure dependence of the convolution and its dependence on the population levels in the neighborhood [3].

3. Child Swing :

This is an example of real life. Swing is a simple model to see the effect on frequency and amplitude with a small change in length. This can be modeled by a mathematical equation as

$$\frac{d^2\delta}{dt^2} + \frac{2}{l}\frac{dl}{dt}\frac{d\delta}{dt} + \sin\delta = 0, \quad t \geq 0,$$

with the angle $\delta(t)$ of the swing.

4. Deformation Elastostatics of a Spherical Shell :

Deformation elastostatics of an isotropic, homogeneous, thin spherical shell of constant thickness with axisymmetric normally distributed surface load. A stress function (ψ) is defined by the following boundary value problem

$$\varepsilon[\varphi'' + (\cot\theta)\varphi' + (v - \cot^2\theta)\varphi] - \frac{1}{\sin\theta}(\cos\alpha - \cos\theta)$$
$$= \varepsilon[vP' + (1+v)P\cot\theta - \frac{1}{\sin\theta}(\delta^2\sin^2\theta)' - v\delta\cos\theta], \quad 0 \leq \theta \leq \frac{\pi}{2},$$
$$\frac{\mu^4}{\varepsilon}[\psi'' + \cot\theta\psi' + \frac{\cos\alpha}{\sin^2\theta}(\sin\alpha - \sin\theta)] - \frac{v}{\sin\theta}(\cos\alpha - \cos\theta) + \frac{\sin\alpha}{\sin\theta}\varphi = \frac{\cos\alpha}{\sin\theta}P,$$
$$\psi(0) = \varphi(0) = \psi(\pi/2) = \varphi(\pi/2) = 0,$$

with

$$P(\theta) = -\int_0^{\theta}(1-\eta\sin\xi)\cos\alpha\sin\xi d\xi, \quad \text{and} \quad \delta = -\sin\alpha(1-\eta\sin\theta),$$

where θ is the angle between the base plane and the meridional tangent, γ be the meridional angle change of the deformed middle surface, $\alpha = \theta\psi$ .Here $\eta(>1)$ is a constant and $v(=0)$ is a typical value. μ and ε are the small perturbation parameters [4].

5. Dynamics of a Network of Two Amplifiers :

The following system of first order differential equations represents the dynamical network of the two amplifiers having delayed outputs

$$\mu u'(t) = -u(t) + g(v(t-1),\gamma),$$
$$\mu v'(t) = -v(t) + g(u(t-1),\gamma),$$

with $g \in C^m(\mathbf{R}\times\mathbf{R})$, $m \geq 3$ and $\mu(>0)$ is a small perturbation parameter [7].

6. Drilling by Laser :

The process of drilling (through a thick block) of a material using a laser is a one-dimensional model. The temperature relative to the ambient condition satisfies the following heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad t > 0, \quad 0 < x < \infty,$$
$$T(x,0,\mu) = 0, \quad \text{as} \quad x \to \infty.$$

The vaporization condition at the bottom of the drill hole at $x = X(t,\mu)$ is

$$T(X(t,\mu),t,\mu) = 1.$$

The speed of the drilling process is controlled by

$$\frac{dT}{dt} = 1 + \mu\frac{dT}{dx}(X(t,\mu),t,\mu), \quad X(0,\mu) = 0.$$

7. Exit Time Problems and Phase Transition Models :

The singularly perturbed boundary value problem

$$\mu y'' + f(x) y'^{q} = 0, \quad 0 < t < T,$$
$$y(0) = C, \quad y(T) = B,$$

where μ is an infinitesimal parameter and $q = 0$ or $1 \le q \le 2$. C, D and T are some standard values. $f(x)$ is a smooth function with the zeros in $[C, D]$. Such problems arise in the study of the exit time problems for stochastic differential equations [22] and in the phase transition models [11].

8. Fokker-Planck Equation :

The following second order differential equation describes the time independent Fokker-Planck equation

$$\mu^2 \frac{d^2u}{dx^2} + a(x)\frac{du}{dx} = 0,$$
$$u(0) = C, \quad u(1) = D,$$

where μ is a small perturbation parameter. The function $a(x)$ describes the gradient field and C and D are the constants [15].

9. Generalized Model of Neuronal Excitation :

The initial value problem for diffusion process using first exit time theory to determine the time of first spike results in the singularly perturbed differential equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\rho^2}{2}\frac{\partial^2 u(x,t)}{\partial x_0^2} + (\gamma_0 - \frac{x}{\eta})\frac{\partial u(x,t)}{\partial x_0} + a_s u(x + a_s, t) + \beta u(x + i_s, t) - (\alpha_s + \beta_s) u(x,t),$$

where the exponential decay cause by the input processes between two consecutive jumps is described by the $\frac{\partial u(x,t)}{\partial t}$. The membrane potential decay exponentially to the resting level with a membrane time constant η. The diffusion moments of Wiener process are γ_0 and ρ. The reaction terms correspond to the inhibitory inputs and superposition of excitatory can be assumed to be Poissonian [24]. Here, the amplitude a_s denotes the excitatory input with an intensity α_s and the amplitude i_s describes the inhibitory input with an intensity β_s. This is a generalization of the Steins model [29, 30].

10. Kinetics of Catalyzed Reaction :

In the model, the concentration of the reactant varies according to the following differential equation

$$\mu u_x = 1 - \frac{2}{xu}(1-u)(\lambda + u), \quad 0 < x \le 1,$$

where $0 < \mu < 1$ is a positive constant and $u(1, \mu) = 0$.

11. Motion of Thin Liquid Film :

The boundary value problems that arises in the theory of thin film flows with gravitational, viscous, and capillary forces is of the form

$$\mu^2 u''' = g(u)u' + f(t,u), \quad a < t < b,$$
$$u(a,u) = C, \quad u'(a,\mu) = D, \quad u(b,\mu) = E,$$

with $\mu^2 = 1/(\mathrm{Re}\, Ca)$, where Ca denotes the Capillary number and Re denotes Reynolds number [16].

12. Motion of a Mass :

The motion of a mass moving in a medium with damping (proportional to the displacement) with either the damping large or the mass small is described by the following boundary value problem [25, 33]

$$\mu u'' + uu' = 0, \quad t \in [0,1],$$
$$u(0) = \alpha, \quad u(1) = \beta.$$

13. Motion of a Sunflower :

The motion of a Sunflower is examined by the following singularly perturbed differential equation [26]

$$\mu y''+ry'+s\sin y(t-\mu)=0, \quad \mu>0,$$
$$y(t)=\phi(t), \ \ t\in[-\mu,0], \ \ y'(0)=x_0,$$

where $y(t)$ denotes the angle of the plant with the vertical [6], r and s are the positive constants and $\mu>0$ is a geotropic reaction time.

14. Optical and Physiological Models :

The mathematical models in biology, optics and physiology are given by following the singularly perturbed differential equation [10, 13, 17, 18, 19]

$$\mu y'(t)+y(t)=g(y(t-1)).$$

Here for the functions like $g(y)=\beta y^{x}(1+y^{\kappa})^{-1}, \ x\ge 0$ and $g(y)=\beta y^{x}e^{-y}$, the equation occurs in physiological models and for $g(y)=\beta_1+\beta_2\sin(\beta_3 y+\beta_4)$, the equation arise in optics.

15. Piston Problem :

The flow of gas in a long, open-ended tube is modeled by the following differential equations

$$u_t+vu_x+\frac{1}{2}(\lambda-1)uv_x=0,$$
$$v_t+vv_x+\frac{2}{\lambda-1}uu_x=0,$$

with the initial and boundary conditions

$$v=0, \ \ u=1, \ \ \text{for} \ \ t=0, \ \ x>\mu x_p(t),$$
$$v=\mu V(t) \ \ \text{on} \ \ x=\mu x_p(t), \ \ t\ge 0,$$

where $x_p(t)=\int_0^t V(t')dt'$ and $V(0)=0$. Here, v denotes the speed of sound along the tube and u is the speed of the sound in the gas.

16. Processing of Metal Sheets :

The processing of metal sheets is modeled by the following singularly perturbed partial differential equation

$$\frac{\partial v_\mu}{\partial t}=\mu\frac{\partial^2 v_\mu}{\partial x^2}+u(f(v_\mu(x,t-\rho)))\frac{\partial v_\mu}{\partial x}+a[g(v_\mu(x,t-\rho))-v_\mu(x,t)],$$

where v_μ denotes the distribution of the temperature in the metal sheet having velocity u and is heated by the source, g. The speed of the controller introduces a fixed delay of length ρ [3].

17. Problem of a Stationary Diffusion Process :

The stationary diffusion process including a reacting substance is defined by the following boundary value problem

$$-\mu u''(t)+g(t,u)=0, \ \ t\in[0,1],$$
$$u(0)=0, \ \ u(1)=0,$$

where $0<\mu \ll 1$, is a small perturbation parameter. The function $g(t,u)$ is sufficiently smooth and $g_u(t,u)>\alpha^2>0, \ \ (t,u)\in[0,1]\times\mathbf{R}$ [31].

18. Projectile Motion with Small Drag :

A projectile moving in a two dimensional (x,z) plane under the action of a drag force proportional to the square of the speed and a gravitational force is provided by the following system of equations

$$\frac{dv}{dt}+1+\mu v\sqrt{u^2+v^2}=0,$$
$$\frac{du}{dt}+\mu u\sqrt{u^2+v^2}=0,$$
$$v=\sin\beta, \ \ u=\cos\beta, \ \ x=z=0, \ \ \text{for all} \ \ t=0, \ \ 0<\beta<\pi/2.$$

Here $(u,v)=\frac{d}{dt}(x,z)$ and β is the angle of projection.

19. Quantum Mechanics :

A quantum mechanics of a particle with a potential energy $V(x)$ is described by an equation

$$-\mu^2 y'' + (V(x)-E)y(x) = 0.$$

Here, E denotes the total energy of the particle. For this problem, the term $V(x)-E = Q(x)$ vanishes at the turning points where $V(x)=E$. The classical orbit of a particle is confined to regions where $V(x)\le E$ [5].

20. Steady State Navier Stokes Equation :

The steady state Navier-Stokes equation is defined as

$$\Delta\phi + \vartheta = 0 \text{ in } G,$$
$$\Delta\phi + \mathrm{Re}(\phi_x\vartheta_y - \phi_y\vartheta_x) = 0 \text{ in } G,$$

where $\mathrm{Re} = 1/\mu$ refers the Reynolds's number with a small perturbation parameter $0<\mu\le 1$. Moreover ϕ and ϑ are prescribed on the boundary ∂G [15].

21. Schordinger Equation :

The one-dimensional Schrodinger equation is given by

$$\mu^2 \frac{d^2\Phi_\mu}{dt^2} + (\eta_\mu - V(t))\Phi_\mu = 0, \quad \|\Phi\| = 1,$$

where the potential energy $V(t)\to+\infty$ as $|t|\to 1$ and $\mu = \frac{h\sqrt{2m}}{2\pi}$ with the mass (m) and the Planck's constant (h). η_μ denotes the energy of the system [8].

22. The Allen-Cahn Equation in Material Science :

The Allen-Cahn equation occurs in material sciences is given by the following system

$$\mu^2\Delta v(x) + v(x) - v^3(x) = 0, \text{ in } \Omega,$$
$$\frac{\partial v(x)}{\partial \nu} = 0, \text{ in } \partial\Omega,$$

where the function $v(x)$ denotes the continuous realization of the phase in a material confined to the region Ω at the point x and ν defines the outer unit normal to boundary $\partial\Omega$ [1].

23. The Orr-Sommerfeld Equation :

The Orr-Sommerfeld equation occurs in the field of fluid dynamics is of the form:

$$\mu\left(\frac{d^2}{dt^2}-\beta^2\right)^2 y = i\left[(U-\xi)\left(\frac{d^2}{dt^2}-\beta^2\right)y - U''y\right],$$
$$y(\pm 1) = 0, \quad y'(\pm 1) = 0,$$

where $\mu = 1/\beta R$ with R, a very high Reynold's number and U denotes the perturbation velocity [2, 14, 20, 23, 28, 32, 34].

24. Van der Pol Equation :

A Van der Pol model with a delayed time $0\le\tau<\pi/2$ was considered by Oliveira. The delayed differential equation is

$$u''(t) - \mu u'(t) + \mu u^2(t-\tau) + u(t) = 0,$$

where $0<\mu \ll 1$ is a small perturbation parameter [9].

REFERENCES

[1] Allen, S. and Cahn, J.W., 1979, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. Acta Metall., 27: 1084-1095.

[2] Amnick, C.J. and McLeod, J.B. 1990. A Singular Perturbation Problem in Needle Crystals, Arch. Rational Mech. Anal., 109(2): 139-171.

[3] Ansari, A. R., Bakr, S. A. and Shishkin, G. I. 2007. A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations. J. Comput. Appl. Math., 205(1): 552-566.

[4] Ascher, U. M., Mattheij, R. M. M. and Russell, R. D. 1988. Numerical solution of boundary value problems for ordinary differential equations. Prentice Hall Series in Computational Mathematics, Prentice Hall Inc., Englewood Cliffs, NJ.

[5] Bender, C. M. and Orszag, S. A. 1978. Advanced mathematical methods for scientists and engineers. McGraw-Hill Book Co., New York, International Series in Pure and Applied Mathematics.

[6] Casal, A. and Somolinos, A. 1982. Forced Ascillations for the Sunflower Equation, Entrainment,Nonlinear Anal., 6: 397-414.

[7] Cen, Y. and Wu, J. 2001. The asymptotic shapes of periodic solutions of a singular delay differential system, J. Differential Equations, 169: 614-632.

[8] De-Jager E.M. and Kpper, T. 1978. The Schrodinger Equation as a Singular Perturbation, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 82(12): 1-11.

[9] De Oliveira, J. C. F. 2002. Oscillations in a van der pol equation with delayed arguments, J. Math. Anal. Appl., 275: 789-803.

[10] Derstine, M. W., Gibbs, H. M., Hopf, F. A. and Kaplan, D. L., 1982. Bifurcation gap in a hybrid optical system. Phys. Rev. A, 26: 3720-3722.

[11] Fife, P. C., 1988, Dynamics of internal layers and diffusive interfaces, CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 53.

[12] Friedrichs, K.O. and Wasow, W., 1946. Singular perturbations of nonlinear oscillations, Duke Math. J. 13: 367–381.

[13] Gibbs, H. M., Hopf, F. A., Kaplan, D. L. and Shoemaker, R. L. 1981. Observation of chaos in optical bistability. Phys. Rev. Lett., 46: 474-477.

[14] Glovangigli, V. 1990. Nonadiabatic Plane Laminar Flames and Their Singular Limits, SIAM Journal on Mathematical Analysis, 21(5): 1305-1325.

[15] Grasman, J. 1979. Small random perturbations of dynamical systems with applications in population genetics. In Asymptotic analysis, Lecture Notes in Math., Springer. Berlin, 711: 158-175.

[16] Howes, F.A. 1983. The Asymptotic Solution of a Class pf Third-Order Boundary Value Problems Arising in the Theory of Thin Film Flows, SIAM Journal on Mathematical Analysis, 43(5): 993-1004.

[17] Ikeda, K. 1979. Multiple-valued stationary state and its instability of the transmitted light by aring cavity system. Opt. Commun., 30: 257-261.

[18] Ikeda, K., Damo, H. and Arauoro, O., 1982. Optical turbulence: Chaotic behaviour of transmitted light from a ring cavity. Phys. Rev. Lett., 49: 709-712.

[19] Ikeda, K., Kondo, K. and Akimoto, O., 1982. Successive higher-harmonic bifurcation in systems with delayed feedback. Phys. Rev. Lett., 49: 1467-1470.

[20] Kellogg, R.B., 1987. Some Singular Perturbation Problems in Renel Models, J. Math. Anal. Appl., 128(1): 214-240.

[21] Li, S., Shishkin, G. I. and Shishkina, L. P.. 2007, Approximation of the solution and its derivative for the singularly perturbed Black-Scholes equation with nonsmooth initial data. Zh. Vychisl. Mat. Mat. Fiz., 47(3): 460-480.

[22] Matkowsky, B. J. 1980. Singular perturbations, stochastic differential equations, and applications.In Singular perturbations and asymptotics (Proc. Adv. Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1980), Academic Press, New York, 45: 109-147.

[23] Miller, J.J.H. 1993. Applications of Advanced Computational Methods for Boundary and Interior Layers, Boole Press, Dublin.

[24] Musila, M. and Lansky, P., 1991. Generalized steins model for anatomically complex neurons. Biosys., 25.

[25] OMalley, Jr.R.E. 1974. Introduction to Singular Perturbations, Academic Press, New York.

[26] Pena, M.L., 1989. Asymptotic Expansion for the Initial Value Problem of the Sunflower Equation, Journal of Mathematical Analysis and Applications, 143(2): 471-479.

[27] Prandtl, L., 1905. Uber Flussigkeits – bewegung bei kleiner Reibung, in: Verhandlungen, III, Int. Math. Kongresses, Tuebner, Leipzig, 484–491.

[28] Sobolev, V.A. 1991. Singular Perturbations in Linearly Quadratic Optimal Control Problems, Automat. Remote Control, 52(1): 180-189.

[29] Stein, R. B. 1965. A theoretical analysis of neuronal variability. Biophy. J., 5: 173-194.

[30] Stein, R. B. 1967. Some models of neuronal variability. Biophy. J., 7: 37-68.

[31] Surla, K., 1997. On Modeling of Semilinear Singularly Perturbed Reaction- Diffusion Problem, Nonlinear Analysis: Theory, Methods and Applications, 30(1): 61-66.

[32] Szmolyan, P. 1989. A Singular Perturbation Analysis of the Transient Semiconductor Device Equations, SIAM Journal on Applied Mathematics, 49(4): 1122-1135.

[33] Wasow, W. 1970. The Capriciousness of Singular Perturbations, Nieuw Arch. Wisk, 18, 190-210.

[34] Wolfe, P. 1993. Deformation of Rods with Small Bending Stiffness, Quarterly Journal of Mechanics and Applied Mathematics, 46(2): 311-330.