# Fibonacci Sequence Generated From Two Dimensional q-difference Operator

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## *Abstract*

*In this paper, we defined generalized Fibonacci sequence using two dimensional q-difference operator and we derive some algebraic identities as it includes its relationship with Fibonacci numbers. Also we derive theorems using inverse two dimensional q-difference operator.*

***Key words:***

*Fibonacci numbers, Two dimensional q-difference operator and Summation solution.*

## 1.Introduction

In 1984, Jerzy Popenda introduced a particular type of difference operator $\Delta_\alpha$ defined on $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, K.S.Miller and Ross introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional derivative operator. Recently, G.Britto Antony Xavier have got the solution of the generalized q-difference equation $\Delta_q^{-t} v(k) = u(k)$, $k \in (-\infty, \infty)$ and q $\neq 1$, in the form

$$\Delta_q^{-t} u(e^k) \Bigg\|_{\frac{e^k}{q^m}}^{e^k} = \sum_{(r)_{1\to t}}^{m} u\left(\frac{e^k}{\prod_{j=1}^{t} q^{r_j}}\right)$$

The authors introduced q-alpha difference operator, which is defined as

$$\Delta_{(q)\alpha} v(e^k) = v(qe^k) - \alpha v(e^k) \tag{1}$$

And then extended to generalized higher order q-alpha difference equation

$$\Delta_{(q_1)\alpha_1}\left(\Delta_{(q_2)\alpha_2}\left(\ldots.\ \Delta_{(q_t)\alpha_t}(v(e^k))\ \ldots.\ \right)\right) = u(e^k), \qquad e^k \in (-\infty, \infty)\,, \tag{2}$$

and obtained formula for finite q-alpha multi-series and finite higher order q-alpha series.

**Definition :1.1**

Let $\gamma_1,\ and\ \gamma_2$ be fixed real's, $e^k \in (-\infty,\infty)$. Then the two-dimensional q $-$ difference operator $\underset{(\gamma_1,\gamma_2)}{\Delta_q}$ is defined as

$$\underset{(\gamma_1,\gamma_2)}{\Delta_q} v(e^k) = v(q^2e^k) - \gamma_1 v(qe^k) - \gamma_2 v(e^k) \qquad (3)$$

and its inverse, denoted by $\underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}}$, is defined as below:

$$\text{if } \underset{(\gamma_1,\gamma_2)}{\Delta_q} v(e^k) = u(e^k), \qquad then\ v(e^k) = \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} u(e^k) \qquad (4)$$

**Remark : 1.2**

When $\gamma_1 = \gamma\ and\ \gamma_2 = 0,\ replacing\ e^k\ by\ \frac{e^k}{q}\ in\ (1)$ we get

$$\Delta_{(q)\gamma} v(e^k) = v(qe^k) - \gamma v(e^k)$$

**Lemma :1.3**

$$If\ q^{2n} - \gamma_1 q^n - \gamma_2 \neq 0\ for\ n = 0,1,2,\dots\ ,\ then\ \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} e^{k^n} = \frac{e^{k^n}}{q^{2n} - \gamma_1 q^n - \gamma_2}$$

$$\text{and } \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} = \frac{1}{1 - \gamma_1 - \gamma_2}$$

Proof :

Replacing $v(e^k)\ by\ e^{k^n}\ in\ (1)\ we\ get,$

$$\underset{(\gamma_1,\gamma_2)}{\Delta_q} e^{k^n} = q^{2n}(e^{k^n}) - \gamma_1 q^n(e^{k^n}) - \gamma_2(e^{k^n})$$

$$e^{k^n} = \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} \left[q^{2n}(e^{k^n}) - \gamma_1 q^n(e^{k^n}) - \gamma_2(e^{k^n})\right]$$

$$\underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} e^{k^n} = \frac{e^{k^n}}{q^{2n} - \gamma_1 q^n - \gamma_2} \qquad (5)$$

again replacing $v(e^k)\ by\ e^{k^0}\ in\ (1)\ we\ get,$

$$\underset{(\gamma_1,\gamma_2)}{\Delta_q} e^{k^0} = q^{2.0}(e^{k^0}) - \gamma_1 q^0(e^{k^0}) - \gamma_2(e^{k^0})$$

$$\text{Then we get } \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} (1) = \frac{1}{1 - \gamma_1 - \gamma_2} \qquad (6)$$

**Lemma :1.4**

$Let \;\; k \in (-\infty, \infty) \;\; and \;\; q \neq 0. \;\; Then \;\; we \;\; have$

$$\Delta^2_{(q)\gamma} v(e^k) = \underset{(\,2\gamma, -\gamma^2)}{\Delta_q} v(e^k)$$

Proof:

From (1) => $\underset{(\,\gamma_1,\gamma_2)}{\Delta_q} v(e^k) = v(q^2 e^k) - \gamma_1 v(qe^k) - \gamma_2 v(e^k)$

Putting $\gamma_1 = 2\gamma \;\; and \;\; \gamma_2 = -\gamma^2$

$$\underset{(\,\gamma_1,\gamma_2)}{\Delta_q} v(e^k) = v(q^2 e^k) - 2\gamma v(qe^k) + \gamma^2 v(e^k)$$

$$\Delta^2_{(q)\gamma} v(e^k) = v(q^2 e^k) - 2\gamma v(qe^k) + \gamma^2 v(e^k)$$

$$\therefore \Delta^2_{(q)\gamma} v(e^k) = \underset{(\,2\gamma, -\gamma^2)}{\Delta_q} v(e^k)$$

## 2. Fibonacci Sequence Using Two – dimensional q-difference operator

In this section, we introduce two dimensional sequence and its sum

**Definition :2.1**

For each pair $(\,\gamma_1,\gamma_2) \in \mathbb{R}^2$, the two dimensional Fibonacci sequence is defined as

$$F_{(\,\gamma_1,\gamma_2)} = \{F_n\}_{n=0}^{\infty}, \tag{7}$$

$where \;\; F_0 = 1, F_1 = \gamma_1 \;\; and \;\; F_n = \gamma_1 F_{n-1} + \gamma_2 F_{n-2} \;\; for \;\; n \geq 2.$

when $\gamma_1 = \gamma_2 = 1, (5) \; become \;\; the \;\; Fibonacci \;\; Sequence.$

**Example :2.2**

$F_{(2,-3)} = \{1, 2, 1, -4, -11, \dots\}$

**Theorem : 2.3**

$Let \; F_n \in F_{(\,\gamma_1,\gamma_2)} \;\; and \;\; e^k \in (-\infty, \infty). \;\; Then \;\; we \;\; have$

$$\sum_{r=0}^{m} F_r \, u\left(\frac{e^k}{q^{r+2}}\right) = \underset{(\,\gamma_1,\gamma_2)}{\Delta_q^{-1}} u(e^k) - F_{m+1} \underset{(\,\gamma_1,\gamma_2)}{\Delta_q^{-1}} u\left(\frac{e^k}{q^{m+1}}\right) - \gamma_2 F_m \underset{(\,\gamma_1,\gamma_2)}{\Delta_q^{-1}} u\left(\frac{e^k}{q^{m+2}}\right) \tag{8}$$

Proof :

$Taking \;\; \underset{(\,\gamma_1,\gamma_2)}{\Delta_q^{-1}} u(e^k) = v(e^k), \;\; \underset{(\,\gamma_1,\gamma_2)}{\Delta_q} v(e^k) = u(e^k) \;\; and \;\; by \;\; (1), we \; write$

$$v(q^2e^k) = u(e^k) + \gamma_1 v(qe^k) + \gamma_2 v(e^k) \tag{9}$$

Replacing $e^k$ by $\frac{e^k}{q}$ in (7) we get,

$$v(qe^k) = u\left(\frac{e^k}{q}\right) + \gamma_1 v(e^k) + \gamma_2 v\left(\frac{e^k}{q}\right) \tag{10}$$

Substituting the value of $v(qe^k)$ in (8), we get

$$v(q^2e^k) = u(e^k) + \gamma_1\left[u\left(\frac{e^k}{q}\right) + \gamma_1 v(e^k) + \gamma_2 v\left(\frac{e^k}{q}\right)\right] + \gamma_2 v\left(\frac{e^k}{q}\right)$$

$$v(q^2e^k) = u(e^k) + \gamma_1 u\left(\frac{e^k}{q}\right) + (\gamma_1^2 + \gamma_2)v(e^k) + \gamma_1\gamma_2 v\left(\frac{e^k}{q}\right) \tag{11}$$

Again replacing $e^k$ by $\frac{e^k}{q}$ in (8) we get,

$$v(e^k) = u\left(\frac{e^k}{q^2}\right) + \gamma_1 v\left(\frac{e^k}{q}\right) + \gamma_2 v\left(\frac{e^k}{q^2}\right)$$

Substituting the value of $v(e^k)$ in (9), we get

$$v(q^2e^k) = u(e^k) + \gamma_1 u\left(\frac{e^k}{q}\right) + (\gamma_1^2 + \gamma_2)\left[u\left(\frac{e^k}{q^2}\right) + \gamma_1 v\left(\frac{e^k}{q}\right) + \gamma_2 v\left(\frac{e^k}{q^2}\right)\right] + \gamma_1\gamma_2 v\left(\frac{e^k}{q}\right)$$

$$v(q^2e^k) = u(e^k) + \gamma_1 u\left(\frac{e^k}{q}\right) + (\gamma_1^2 + \gamma_2)u\left(\frac{e^k}{q^2}\right) + \{\gamma_1(\gamma_1^2 + \gamma_2) + \gamma_1\gamma_2\}v\left(\frac{e^k}{q}\right)$$
$$+ \gamma_2(\gamma_1^2 + \gamma_2)v\left(\frac{e^k}{q^2}\right) \tag{12}$$

Since $F_n \in F_{(\gamma_1,\gamma_2)}$, we get

$$v(q^2e^k) = F_0 u(e^k) + F_1 u\left(\frac{e^k}{q}\right) + F_2 u\left(\frac{e^k}{q^2}\right) + F_3 v\left(\frac{e^k}{q}\right) + \gamma_2 F_2 v\left(\frac{e^k}{q^2}\right) \tag{13}$$

Proceeding like this we, arrive

$$v(q^2e^k) = F_0 u(e^k) + F_1 u\left(\frac{e^k}{q}\right) + \ \ldots + F_m u\left(\frac{e^k}{q^m}\right) + F_{m+1}\left(\frac{e^k}{q^{m-1}}\right) + \gamma_2 F_m v\left(\frac{e^k}{q^m}\right) \tag{14}$$

$$v(e^k) = F_0 u\left(\frac{e^k}{q^2}\right) + F_1 u\left(\frac{e^k}{q^3}\right) + \ \ldots + F_m u\left(\frac{e^k}{q^{m+2}}\right) + F_{m+1}\left(\frac{e^k}{q^{m+1}}\right) + \gamma_2 F_m v\left(\frac{e^k}{q^{m+2}}\right)$$

$$\sum_{r=0}^{m} F_r\, u\left(\frac{e^k}{q^{r+2}}\right) = \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} u(e^k) - F_{m+1} \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} u\left(\frac{e^k}{q^{m+1}}\right) - \gamma_2 F_m \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} u\left(\frac{e^k}{q^{m+2}}\right)$$

**Corollary : 2.4**

$Assume\ \ that\ \ \gamma_1 + \gamma_2 \neq 1\ \ and\ \ F_n \in F_{(\gamma_1,\gamma_2)}.\ Then$ we have

$$\sum_{r=0}^{m} F_r = \frac{1 - F_{m+1} - \gamma_2 F_m}{1 - \gamma_1 - \gamma_2}$$

Proof :

replacing $u(e^k)$ by $e^{k^0}$ in (8).

$$\sum_{r=0}^{m} F_r = \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} - F_{m+1} \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} - \gamma_2 F_m \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}}$$

$$\sum_{r=0}^{m} F_r = \underset{(\gamma_1,\gamma_2)}{\Delta_q^{-1}} [1 - F_{m+1} - \gamma_2 F_m]$$

$$\sum_{r=0}^{m} F_r = \frac{[1 - F_{m+1} - \gamma_2 F_m]}{1 - \gamma_1 - \gamma_2}$$

**3. Two –Dimensional q Multi – Series**

In this section, we obtain formula for sum of q-multi series.

**Theorem : 3.1**

Let $0 \neq q_{i,}\ \ k \in (-\infty,\infty)\ \ and\ \ F_n \in F_{(\gamma_1,\gamma_2)}.\ Then$

$$\sum_{i=1}^{t-1} \sum_{(r)_{1\to i}}^{m} \prod_{j=1}^{i} F_{r_j} \underset{(\gamma_1,\gamma_2)}{\overset{\Delta^{-1}}{q_{i+1\to t}}} \left\{ F_{m_{i+1}+1} u\left( \frac{\prod_{p=i+1}^{t-1} q_p^2 e^k}{\prod_{p=1}^{i} q_p^{r_p}\, q_{i+1}^{m_{i+1}+1}} \right) + \gamma_2 F_{m_{i+1}} u\left( \frac{\prod_{p=i+1}^{t-1} q_p^2 e^k}{\prod_{p=1}^{i} q_p^{r_p}\, q_{i+1}^{m_{i+1}+2}} \right) \right\}$$
$$+ \sum_{(r)_{1\to i}}^{m} \prod_{i=1}^{t} F_{r_j} u\left( \frac{e^k}{\prod_{i=1}^{t} q_i^{r_i} q_t^2} \right)$$
$$= \underset{(\gamma_1,\gamma_2)}{\overset{\Delta^{-1}}{q_{1\to t}}} \left\{ u\left( \prod_{p=1}^{t-1} q_p^2 e^k \right) - F_{m_1+1} u\left( \frac{\prod_{p=1}^{t-1} q_p^2 e^k}{q_1^{m_1+1}} \right) - \gamma_2 F_{m_1} u\left( \frac{\prod_{p=1}^{t-1} q_p^2 e^k}{q_1^{m_1+2}} \right) \right\}$$

Proof :

Replace q, m, r by $q_2, m_2\ , r_2$ in (3.6), we get

$$\sum_{r_2=0}^{m_2} F_{r_2} u\left( \frac{e^k}{q_2^{r_2+2}} \right) = \underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}} u(e^k) - F_{m_2+1} \underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}} u\left( \frac{e^k}{q_2^{m_2+1}} \right) - \gamma_2 F_{m_2} \underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}} u\left( \frac{e^k}{q_2^{m_2+2}} \right)$$

…… (3.13)

Replace $e^k\ \ by\ \ \frac{e^k}{q_1^{r_1}}\ \ and\ \ multiplying\ by\ F_{r_1}\ \ for\ \ r_1 = 1,2,3,\dots\ m_1\ \ in\ \ (3.13)$

$$F_{r_1}\sum_{r_2=0}^{m_2} F_{r_2}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{r_2+2}}\right)$$
$$= \frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1}}\right) - F_{m_2+1}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{m_2+1}}\right)$$
$$-\gamma_2\, F_{m_2}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{m_2+2}}\right) \qquad (3.14)$$

Summing (3.14) for $r_1 = 1, 2, \ldots\ldots m_1$ we obtain

$$\sum_{r_1=0}^{m_1} F_{r_1}\sum_{r_2=0}^{m_2} F_{r_2}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{r_2+2}}\right)$$
$$= F_{r_1}\left\{\sum_{r_1=0}^{m_1}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1}}\right)\right.$$
$$\left. -\sum_{r_1=0}^{m_1} F_{m_2+1}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{m_2+1}}\right) - \sum_{r_1=0}^{m_1}\gamma_2 F_{m_2}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{m_2+2}}\right)\right\}$$

Using (3.6) the above expression becomes

$$\sum_{r_1=0}^{m_1} F_{r_1}\sum_{r_2=0}^{m_2} F_{r_2}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{r_2+2}}\right)$$
$$= \frac{\Delta_{q_1}^{-1}}{(\gamma_1,\gamma_2)}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u(q_1^2 e^k) - F_{m_1+1}\frac{\Delta_{q_1}^{-1}}{(\gamma_1,\gamma_2)}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{q_1^2 e^k}{q_1^{m_1+1}}\right)$$
$$-\frac{\Delta_{q_1}^{-1}}{(\gamma_1,\gamma_2)}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\gamma_2\; F_{m_1} u\left(\frac{q_1^2 e^k}{q_1^{m_1+2}}\right) - \sum_{r_1=0}^{m_1} F_{r_1}\; F_{m_2+1}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{m_2+1}}\right)$$
$$-\sum_{r_1=0}^{m_1} F_{r_1}\gamma_2 F_{m_2}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_1^{r_1} q_2^{m_2+2}}\right) \qquad (3.15)$$

Replacing the $q_1, m_1, r_1, q_2, m_2, r_2$ *by* $q_2, m_2, r_2\ q_3, m_3, r_3$ *in* (3.15)

$$\sum_{r_2=0}^{m_2} F_{r_2}\sum_{r_3=0}^{m_3} F_{r_3}\, u\left(\frac{e^k}{q_2^{r_2} q_3^{r_3+2}}\right)$$
$$= \frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\frac{\Delta_{q_3}^{-1}}{(\gamma_1,\gamma_2)}\, u(q_2^2 e^k) - F_{m_2+1}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\frac{\Delta_{q_3}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{q_2^2 e^k}{q_2^{m_2+1}}\right) - \gamma_2\, F_{m_2}\frac{\Delta_{q_2}^{-1}}{(\gamma_1,\gamma_2)}\frac{\Delta_{q_3}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{q_2^2 e^k}{q_2^{m_1+2}}\right)$$
$$-\sum_{r_2=0}^{m_2} F_{r_1}\; F_{m_3+1}\frac{\Delta_{q_3}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_2^{r_2} q_3^{m_3+1}}\right)$$
$$-\sum_{r_2=0}^{m_2} F_{r_2}\gamma_2 F_{m_3}\frac{\Delta_{q_3}^{-1}}{(\gamma_1,\gamma_2)}\, u\left(\frac{e^k}{q_2^{r_2} q_3^{m_3+2}}\right) \qquad (3.16)$$

Again replace $e^k$ $by$ $\frac{e^k}{q_1^{r_1}}$ $and\ multiplying\ by\ F_{r_1}\ for\ r_1 = 1,2,3,\ldots\ m_1\ in\ (3.16),$

We get

$$F_{r_1}\sum_{r_2=0}^{m_2}F_{r_2}\sum_{r_3=0}^{m_3}F_{r_3}\,u\left(\frac{e^k}{q_1^{r_1}q_2^{r_2}q_3^{m_3+2}}\right)$$
$$= F_{r_1}\left\{\underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}u\left(\frac{q_2^2e^k}{q_1^{r_1}}\right) - F_{m_2+1}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}u\left(\frac{q_2^2e^k}{q_1^{r_1}q_2^{m_2+1}}\right)\right.$$
$$- \gamma_2F_{m_2}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}u\left(\frac{q_2^2e^k}{q_1^{r_1}q_2^{m_1+2}}\right) - \sum_{r_2=0}^{m_2}F_{r_1}\ F_{m_3+1}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}u\left(\frac{e^k}{q_1^{r_1}q_2^{r_2}q_3^{m_3+1}}\right)$$
$$\left.- \sum_{r_2=0}^{m_2}F_{r_2}\gamma_2F_{m_3}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}u\left(\frac{e^k}{q_1^{r_1}q_2^{r_2}q_3^{m_3+2}}\right)\right\}$$

$$\sum_{r_1=0}^{m_1}F_{r_1}\sum_{r_2=0}^{m_2}F_{r_2}\sum_{r_3=0}^{m_3}F_{r_3}\,u\left(\frac{e^k}{q_1^{r_1}q_2^{r_2}q_3^{m_3+2}}\right)$$
$$= \underset{(\gamma_1,\gamma_2)}{\Delta_{q_1}^{-1}}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}\left\{u(q_1^2q_2^2e^k) - F_{m_1+1}u\left(\frac{q_1^2q_2^2e^k}{q_1^{m_1+1}}\right) - \gamma_2\ F_{m_1}u\left(\frac{q_1^2q_2^2e^k}{q_1^{m_1+2}}\right)\right\}$$
$$- \sum_{r_1=0}^{m_1}F_{r_1}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_2}^{-1}}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}\left\{F_{m_2+1}u\left(\frac{q_2^2e^k}{q_1^{r_1}q_2^{m_2+1}}\right) + \gamma_2F_{m_2}u\left(\frac{q_2^2e^k}{q_1^{r_1}q_2^{m_1+2}}\right)\right\}$$
$$- \sum_{r_1=0}^{m_1}F_{r_1}\sum_{r_2=0}^{m_2}F_{r_1}\left\{F_{m_3+1}\underset{(\gamma_1,\gamma_2)}{\Delta_{q_3}^{-1}}u\left(\frac{e^k}{q_1^{r_1}q_2^{r_2}q_3^{m_3+1}}\right) + \gamma_2F_{m_3}u\left(\frac{e^k}{q_1^{r_1}q_2^{r_2}q_3^{m_3+2}}\right)\right\}$$

Proceeding like this we get the proof of the theorem

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