A common fixed point theorem in complex valued A_b-metric space

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Abstract: In this paper, we prove a common fixed point theorem for two self mappings in complex valued A_b-metric space. Our result (with some modifications) generalizes a common fixed point result in complex valued S_b-metric space by N. Priyobarta et al. [10] which is already a generalization of a result by Nabil M. Mlaiki [11].
***Keywords*:** Complex valued S_b-metric space, complex valued A_b-metric space and common fixed point.

1. Introduction

In 2011, Azam et al. [1] introduced the concept of complex valued metric space as a generalization of metric space and proved some fixed point results for a pair of mappings for a contraction condition satisfying a rational expression. After this, many authors have generalized the complex valued metric space in various directions. In 2013, K. Rao et al. [5] introduced complex valued b-metric space as a generalization of complex valued metric space. In 2014, Nabil M. Mlaiki [11] introduced complex valued S-metric space and proved some common fixed point results. Then in 2017, N. Priyobarta et al. [10] extended complex valued S-metric space to complex valued S_b-metric space and proved some fixed point results including a common fixed point result as a generalization of a result by Nabil M. Mlaiki [11]. Recently K. Anthony Singh and M. R. Singh [4] introduced complex valued A_b-metric space as further generalization of complex valued metric space and proved some fixed point results. Complex valued A_b-metric space can also be looked upon as an extension of A_b-metric space introduced by Manoj Ughade et al. [7].
The aim of this paper is to present a common fixed point result in complex valued A_b-metric space. Our result (with some modifications) generalizes a result of N. Priyobarta et al. [10].

2. Preliminaries

In this section, we recall some properties of A-metric space, A_b-metric space, complex valued metric space, complex valued b-metric space, complex valued S-metric space, complex valued S_b-metric space and complex valued A_b-metric space.
Definition 2.1. [8] Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is called an A-metric on X if for any $x_i, a \in X$, $i = 1,2, \dots, n$, the following conditions hold:

(A1) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$,

(A2) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \dots = x_{n-1} = x_n$,

(A3)
$$\begin{aligned} A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq\ & [A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ & + A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ & + A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) \\ & \dots \dots \dots \\ & + A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\ & + A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]. \end{aligned}$$

The pair (X, A) is called an A-metric space.
Definition 2.2. [7] Let X be a nonempty set and $b \geq 1$ be a given real number. A function $A: X^n \to [0, \infty)$ is called an A_b –metric on X if for any $x_i, a \in X$, $i = 1,2, \dots, n$, the following conditions hold:

(A_b1) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$,

(A_b2) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1= x_2 = x_3 = \dots = x_{n-1} = x_n$,

(A_b3)
$$\begin{aligned} A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq\ & b[A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ & + A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ & + A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) \\ & \dots \dots \dots \\ & + A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\ & + A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]. \end{aligned}$$

The pair (X, A) is called an A_b-metric space.
Note: A_b-metric space is more general than A-metric space. Moreover, A-metric space is a special case of A_b-metric space with b =1.
Example 2.3. [7] Let $X = [1, +\infty)$. Define $A_b : X^n \to [0, +\infty)$ by

$$A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} \left|x_i - x_j\right|^2$$

for all $x_i \in X$, $i=1, 2, \dots, n$.
Then (X, A_b) is an A_b-metric space with $b = 2 > 1$.
The concept of complex valued metric space was initiated by Azam et al. [1]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.
It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(C_1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(C_2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(C_3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(C_4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

Particularly, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (C_2), (C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:

1. If $a, b \in \mathbb{R}$ with $a \le b$, then $az \precsim bz$ for all $0 \precsim z \in \mathbb{C}$.
2. If $z_1 \precsim z_2$, then $az_1 \precsim az_2$ for all $0 \le a \in \mathbb{R}$.
3. If $0 \precsim z_1 \precsim z_2$, then $|z_1| \le |z_2|$.
4. If $0 \precsim z_1 \lnsim z_2$, then $|z_1| < |z_2|$.
5. If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 2.4. [1] Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$, the following conditions are satisfied:
(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,y) \precsim d(x,z) + d(z,y)$.
The pair (X, d) is called a complex valued metric space.

Definition 2.5. [5] Let X be a nonempty set and let $s \ge 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$, the following conditions are satisfied :
(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,y) \precsim s[d(x,z) + d(z,y)]$.
The pair (X, d) is called a complex valued b-metric space.

Definition 2.6. [11] Let X be a nonempty set and $\mathbb{C}$ the set of all complex numbers. A complex valued S-metric on X is a function $S : X^3 \to \mathbb{C}$ that satisfies the following conditions, for all $x, y, z, t \in X$:
(i) $0 \precsim S(x, y, z)$,
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,
(iii) $S(x, y, z) \precsim S(x, x, t) + S(y, y, t) + S(z, z, t)$.
The pair (X, S) is called a complex valued S-metric space.

Definition 2.7. [10] Let X be a nonempty set and $b \ge 1$ be a given real number. Suppose that a mapping $S : X^3 \to \mathbb{C}$ satisfies:
(CS_b1): $0 \prec S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,
(CS_b2): $S(x, y, z) = 0$ if and only if $x = y = z$,
(CS_b3): $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$,
(CS_b4): $S(x, y, z) \precsim b(S(x, x, a) + S(y, y, a) + S(z, z, a))$ for all $x, y, z, a \in X$.
Then, S is called a complex valued S_b-metric on X and (X, S) is called a complex valued S_b-metric space.

Definition 2.8. [4] Let X be a nonempty set and $b \ge 1$ be a given real number. Suppose that a mapping $A : X^n \to \mathbb{C}$ satisfies for all x_i , $a \in X$, $i = 1, 2, \ldots, n$:
(CA_b1) $0 \precsim A(x_1, x_2, \ldots, x_n)$,
(CA_b2) $A(x_1, x_2, \ldots, x_n) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_n$,
(CA_b3)
$$\begin{aligned} A(x_1, x_2, \ldots, x_{n-1}, x_n) \precsim b[&A(x_1, x_1, \ldots, (x_1)_{n-1}, a) \\ &+ A(x_2, x_2, \ldots, (x_2)_{n-1}, a) \\ &\ldots \ldots \ldots \\ &+ A(x_{n-1}, x_{n-1}, \ldots, (x_{n-1})_{n-1}, a) \\ &+ A(x_n, x_n, \ldots, (x_n)_{n-1}, a)]. \end{aligned}$$
Then A is called a complex valued A_b-metric on X and the pair (X, A) is called a complex valued A_b-metric space.

Example 2.9. [4] Let $X = \mathbb{R}$ and A: $X^n \to \mathbb{C}$ be such that
$A(x_1, x_2, \ldots, x_{n-1}, x_n) = (\alpha + i\beta) A_*(x_1, x_2, \ldots, x_{n-1}, x_n)$,
where $\alpha, \beta \ge 0$ are constants and A_* is an A_b-metric on X. Then A is a complex valued A_b-metric on X. As a particular case, we have the following example of complex valued A_b-metric on X.

The mapping $A : X^3 \to \mathbb{C}$ defined by $A(x_1, x_2, x_3) = e^{i\theta}(|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2)$, $\theta \in \left[0, \frac{\pi}{2}\right]$, is a complex valued A_b-metric on $X = \mathbb{R}$ with $b = 2$ and $n = 3$.

Definition 2.10. [4] A complex valued A_b-metric space (X, A) is said to be symmetric if
$$A(x_1, x_1, \ldots, (x_1)_{n-1}, x_2) = A(x_2, x_2, \ldots, (x_2)_{n-1}, x_1)$$
for all $x_1, x_2 \in X$.

Definition 2.11. [4] Let (X, A) be a complex valued A_b-metric space.
(i) A sequence $\{x_p\}$ in X is said to be complex valued A_b-convergent to x if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $A(x_p, \ldots, x_p, x) \prec a$ or $A(x, \ldots, x, x_p) \prec a$ for all $p \ge k$ and is denoted by $\lim_{p\to\infty} x_p = x$ or $x_p \to x$ as $p \to \infty$.
(ii) A sequence $\{x_p\}$ in X is called complex valued A_b-Cauchy if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $A(x_p, \ldots, x_p, x_q) \prec a$ for each $p, q \ge k$.
(iii) If every complex valued A_b-Cauchy sequence is complex valued A_b-convergent in X, then (X, A) is said to be complex valued A_b-complete.

Lemma 2.12. [4] *Let (X, A) be a complex valued A_b-metric space and let $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is complex valued A_b-convergent to x if and only if $|A(x_p, \ldots, x_p, x)| \to 0$ as $p \to \infty$ or $|A(x, \ldots, x, x_p)| \to 0$ as $p \to \infty$.*

Lemma 2.13. [4] *Let (X, A) be a complex valued A_b-metric space and $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is complex valued A_b-Cauchy sequence if and only if $|A(x_p, \ldots, x_p, x_q)| \to 0$ as $p, q \to \infty$.*

Lemma 2.14. [4] *Let (X, A) be a complex valued A_b-metric space. Then $A(x, x, \ldots, x, y) \precsim bA(y, y, \ldots, y, x)$, for all $x, y \in X$.*
Theorem 2.15. [10] *Let (X, S) be a complete complex valued S_b-metric space and f, g be two self mappings on X satisfying the following contraction condition:*

$$S(fx, fx, gy) \precsim \alpha S(x,x,y) + \frac{\beta S(x,x,fx)S(y,y,gy)}{b\left(2S(x,x,gy)+S(y,y,fx)+S(x,x,y)\right)}$$

for all $x, y \in X$ such that $x \neq y$, $S(x, x, gy) + S(y, y, fx) + S(x, x, y) \neq 0$ where α, β are two nonnegative real numbers with $\alpha+\beta<1$ or $S(fx, fx, gy)= 0$ if $S(x, x, gy) + S(y, y, fx) + S(x, x, y) = 0$. Then f, g have a unique common fixed point.

Note: In the statement of the above theorem, we have some observations. If $x \neq y$, then $S(x, x, y) \neq 0$ and so $S(x, x, gy) + S(y, y, fx) + S(x, x, y) \neq 0$. Therefore the condition $S(x, x, gy) + S(y, y, fx) + S(x, x, y) \neq 0$ is not necessary. Also in the second case, if $S(x, x, gy) + S(y, y, fx) + S(x, x, y) = 0$, then $S(x, x, y) = 0$, $S(y, y, fx) = 0$ and $S(x, x, gy) = 0$. And this implies that $fx = gy = x = y$ and therefore $S(fx, fx, gy)= 0$. Thus the second case is an obvious implication and not a condition.

3. Main Result

We now state and prove our main result.
Our Theorem is a generalization of Theorem 2.15. with some modifications in the light of the Note above. Also, to compensate for the symmetry condition in complex valued S_b-metric space which is required in the proof of the Theorem, we make our space symmetric.
Theorem 3.1. *Let (X, A) be a complete complex valued A_b-metric space which is symmetric and f, g be two self mappings on X satisfying the following contraction condition*

$$A(fx, fx, \ldots, fx, gy) \precsim \alpha A(x,x,\ldots,x,y) + \frac{\beta A(x,x,\ldots,x,fx)A(y,y,\ldots,y,gy)}{b[(n-1)A(x,x,\ldots,x,gy)+A(y,y,\ldots,y,fx)+A(x,x,\ldots,x,y)]} \tag{1}$$

for all $x, y \in X$ such that $x \neq y$, where α, β are two nonnegative real numbers with $b(\alpha+\beta)<1$. Then f, g have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. And, let a sequence $\{x_p\}$ in X be defined as $x_{2p+1} = fx_{2p}$ and $x_{2p+2} = gx_{2p+1}$, p = 0, 1, 2, 3, ... And we suppose that $x_{2p} \neq x_{2p+1}$, $x_{2p+1} \neq x_{2p+2}$ for any $p \geq 0$. Then, from (1) we have

$$\begin{aligned}
A(x_{2p+1}, x_{2p+1}, \ldots, x_{2p+1}, x_{2p+2}) &= A(fx_{2p}, fx_{2p}, \ldots, fx_{2p}, gx_{2p+1}) \\
&\precsim \alpha A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right) + \\
&\quad \frac{\beta A\left(x_{2p},x_{2p},\ldots,x_{2p},fx_{2p}\right)A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},gx_{2p+1}\right)}{b\left[(n-1)A\left(x_{2p},x_{2p},\ldots,x_{2p},gx_{2p+1}\right)+A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},fx_{2p}\right)+A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right]} \\
&= \alpha A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right) + \\
&\quad \frac{\beta A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}\right)}{b\left[(n-1)A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+2}\right)+A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right]}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left|A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}\right)\right| &\leq \alpha\left|A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right| + \\
&\quad \frac{\beta\left|A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right|\left|A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}\right)\right|}{b\left|(n-1)A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+2}\right)+A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right|}
\end{aligned} \tag{2}$$

By the symmetry of X and CA_b3, we have

$$\begin{aligned}
\left|A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}\right)\right| &= \left|A\left(x_{2p+2},x_{2p+2},\ldots,x_{2p+2},x_{2p+1}\right)\right| \\
&\leq b\left|(n-1)A\left(x_{2p+2},x_{2p+2},\ldots,x_{2p+2},x_{2p}\right)+A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p}\right)\right| \\
&= b\left|(n-1)A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+2}\right)+A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right|
\end{aligned}$$

Therefore, from (2) we have

$$\begin{aligned}
\left|A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}\right)\right| &\leq \alpha\left|A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right| \\
&\quad +\beta\left|A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right| \\
&= (\alpha+\beta)\left|A\left(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}\right)\right|
\end{aligned} \tag{3}$$

Similarly, using the symmetry of X, we get

$$\left|A\left(x_{2p+2},x_{2p+2},\ldots,x_{2p+2},x_{2p+3}\right)\right| \leq (\alpha+\beta)\left|A\left(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}\right)\right| \tag{4}$$

Combining (3) and (4),we get

$\left|A\left(x_p,x_p,\ldots,x_p,x_{p+1}\right)\right| \le k\left|A\left(x_{p-1},x_{p-1},\ldots,x_{p-1},x_p\right)\right|$ for all $p\in N$, where $k=\alpha+\beta<\frac{1}{b}<1.$ (5)

By repeatedly applying (5), we get

$$\begin{aligned}\left|A\left(x_p,x_p,\ldots,x_p,x_{p+1}\right)\right| &\le k\left|A\left(x_{p-1},x_{p-1},\ldots,x_{p-1},x_p\right)\right|\\ &\le k^2\left|A\left(x_{p-2},x_{p-2},\ldots,x_{p-2},x_{p-1}\right)\right|\\ &\quad \ldots\ \ldots\ \ldots\\ &\le k^p\left|A\left(x_0,x_0,\ldots,x_0,x_1\right)\right|\end{aligned}$$

Thus for any $p<q,\ p,\ q\in N$, we have

$$\begin{aligned}&\left|A\left(x_p,x_p,\ldots,x_p,x_q\right)\right| \le (n-1)b\left|A\left(x_p,x_p,\ldots,x_p,x_{p+1}\right)\right| + b\left|A\left(x_{p+1},x_{p+1},\ldots,x_{p+1},x_q\right)\right|\\ &\le b(n-1)\left|A\left(x_p,x_p,\ldots,x_p,x_{p+1}\right)\right| + b^2(n-1)\left|A\left(x_{p+1},x_{p+1},\ldots,x_{p+1},x_{p+2}\right)\right|\\ &\quad +\ldots+b^{q-p-1}(n-1)\left|A\left(x_{q-2},x_{q-2},\ldots,x_{q-2},x_{q-1}\right)\right| + b^{q-p-1}\left|A\left(x_{q-1},x_{q-1},\ldots,x_{q-1},x_q\right)\right|\\ &\le b(n-1)\left|A\left(x_p,x_p,\ldots,x_p,x_{p+1}\right)\right| + b^2(n-1)\left|A\left(x_{p+1},x_{p+1},\ldots,x_{p+1},x_{p+2}\right)\right|\\ &\quad +\ldots+b^{q-p-1}(n-1)\left|A\left(x_{q-2},x_{q-2},\ldots,x_{q-2},x_{q-1}\right)\right|\\ &\quad +b^{q-p}(n-1)\left|A\left(x_{q-1},x_{q-1},\ldots,x_{q-1},x_q\right)\right|\\ &\le (n-1)\left[bk^p+b^2k^{p+1}+\ldots+b^{q-p-1}k^{q-2}+b^{q-p}k^{q-1}\right]\left|A\left(x_0,x_0,\ldots,x_0,x_1\right)\right|\\ &\le (n-1)\left[(bk)^p+(bk)^{p+1}+\ldots+(bk)^{q-2}+(bk)^{q-1}\right]\left|A\left(x_0,x_0,\ldots,x_0,x_1\right)\right|\\ &\le \frac{(n-1)(bk)^p}{1-bk}\left|A\left(x_0,x_0,\ldots,x_0,x_1\right)\right|\\ &\Rightarrow \left|A\left(x_p,x_p,\ldots,x_p,x_q\right)\right| \le \frac{(n-1)(bk)^p}{1-bk}\left|A\left(x_0,x_0,\ldots,x_0,x_1\right)\right| \to 0 \quad as \quad p,q\to\infty.\end{aligned}$$

Hence $\{x_p\}$ is a complex valued A_b-Cauchy sequence.

Since X is complete, the sequence $\{x_p\}$ converges to some $u\in X$. We show that u is the unique common fixed point of f and g.

Let us assume that $f(u)\ne u$. Then $\left|A(fu, fu,\ldots, fu,u)\right|>0.$

Now we have

$$\begin{aligned}A(fu, fu,\ldots, fu, u) &\precsim (n-1)bA(fu, fu,\ldots, fu,x_{2p+2})+bA(u,u,\ldots,u,x_{2p+2})\\ &=(n-1)bA(fu, fu,\ldots, fu,gx_{2p+1})+bA(u,u,\ldots,u,x_{2p+2})\\ &\precsim (n-1)b\alpha A(u,u,\ldots,u,x_{2p+1})\\ &\quad+\frac{(n-1)b\beta A(u,u,\ldots,u, fu)A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},gx_{2p+1})}{b\left[(n-1)A(u,u,\ldots,u,gx_{2p+1})+A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1}, fu)+A(u,u,\ldots,u,x_{2p+1})\right]}\\ &\quad+bA\left(u,u,\ldots,u,x_{2p+2}\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow \left|A(fu, fu,\ldots, fu,u)\right| &\le (n-1)b\alpha\left|A(u,u,\ldots,u,x_{2p+1})\right|\\ &\quad+\frac{(n-1)\beta\left|A(u,u,\ldots,u, fu)\right|\left|A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2})\right|}{\left|(n-1)A(u,u,\ldots,u,x_{2p+2})+A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1}, fu)+A(u,u,\ldots,u,x_{2p+1})\right|}\\ &\quad+b\left|A(u,u,\ldots,u,x_{2p+2})\right|\\ &\quad\to 0 \quad as \quad p\to\infty.\end{aligned}$$

This is a contradiction to our assumption about $A(fu, fu,\ldots, fu,u)$.

Therefore, we must have $fu = u$. Similarly we can show that $gu = u$. Therefore, u is a common fixed point of f and g.

And to show the uniqueness of the common fixed point of f and g, let $v\in X$ be another common fixed point of f and g. And let us assume that $u\ne v$.

Then we have

$$\begin{aligned}A(u,u,\ldots,u,v) &= A(fu, fu,\ldots, fu,gv)\\ &\precsim \alpha A(u,u,\ldots,u,v)+\frac{\beta A(u,u,\ldots,u, fu)A(v,v,\ldots,v,gv)}{b\left[(n-1)A(u,u,\ldots,u,gv)+A(v,v,\ldots,v, fu)+A(u,u,\ldots,u,v)\right]}\\ &=\alpha A(u,u,\ldots,u,v)\end{aligned}$$

$$\Rightarrow \left|A(u,u,\ldots,u,v)\right| \le \alpha\left|A(u,u,\ldots,u,v)\right| < \left|A(u,u,\ldots,u,v)\right|.$$

This is a contradiction. Therefore, we must have $u = v$.

Hence, f and g have a unique common fixed point.

Corollary 3.2. *Let (X, A) be a complete complex valued A_b-metric space which is symmetric and f be a self mapping on X satisfying the following contraction condition*

$$A(fx, fx, \ldots, fx, fy) \precsim \alpha A(x, x, \ldots, x, y) + \frac{\beta A(x, x, \ldots, x, fx) A(y, y, \ldots, y, fy)}{b[(n-1)A(x, x, \ldots, x, fy) + A(y, y, \ldots, y, fx) + A(x, x, \ldots, x, y)]}$$

for all $x, y \in X$ such that $x \neq y$, where α, β are two nonnegative real numbers with $b(\alpha+\beta) < 1$. Then f has a unique fixed point in X.

Proof: Follows from the proof of Theorem 3.1. by taking $g = f$.

Corollary 3.3. *Let (X, A) be a complete complex valued A_b-metric space which is symmetric and f be a self mapping on X satisfying for some positive integer m, the following contraction condition*

$$A(f^m x, f^m x, \ldots, f^m x, f^m y) \precsim \alpha A(x, x, \ldots, x, y) + \frac{\beta A(x, x, \ldots, x, f^m x) A(y, y, \ldots, y, f^m y)}{b[(n-1)A(x, x, \ldots, x, f^m y) + A(y, y, \ldots, y, f^m x) + A(x, x, \ldots, x, y)]}$$

for all $x, y \in X$ such that $x \neq y$, where α, β are two nonnegative real numbers with $b(\alpha+\beta) < 1$. Then f has a unique fixed point in X.

Proof: From Corollary 3.2., we have f^m has a unique fixed point $u \in X$. And we have $f(f^m u) = fu$ i.e. $f^m(fu) = fu$, which means that fu is a fixed point of f^m. And the uniqueness of the fixed point of f^m implies $f(u) = u$. Therefore, u is a fixed point of f. Further to show the uniqueness of the fixed point of f we easily see that a fixed point of f is also a fixed point of f^m. And the uniqueness of the fixed point of f^m implies the fixed point of f is also unique.

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