The crossing number of the Cartesian product of paths with complete graphs

Authors: 1) K. Venkata Narayana, MSc, FGDA 2) R.V.T.R. Manikyamba Vegulla, MSc 3) Grandhi Suresh, MSc

Abstract

In this paper, we determine the crossing number of $K_n \setminus e$ by the construction method for $m \leq 12$ and apply the zip product to obtain that $cr(K_{n}Q_{m}) = (n - 1)cr(K_{n+2} \setminus e) + 2cr(K_{m+1})$ for $n \geq 1$. Furthermore, we have

$$cr(K_{n}Q_{m}) = \frac{1}{4} \begin{bmatrix} \frac{m+1}{2} & \frac{m-1}{2} & \frac{m-2}{2} & \frac{m+4}{2} & \frac{m-4}{2} \\ \frac{m+1}{2} & \frac{m-1}{2} & \frac{m-2}{2} & \frac{m+4}{2} & \frac{m-4}{2} \end{bmatrix}$$

for $n \geq 1$, $1 \leq m \leq 10$, which is consistent with Zheng’s conjecture for the crossing number of $K_{n}Q_{m}$.

Keywords: Crossing number, Cartesian product, Zip product, Complete graph

1. Introduction

All graphs considered here are simple, finite and undirected and are also connected. For graph theory terminology not defined here, we direct the reader to [7]. A drawing of a graph $G=(V,E)$ is a mapping $\phi$ that assigns to each vertex in $V$ a distinct point in the plane and to each edge $uv$ in $E$ a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $\phi(u)$ and $\phi(v)$, without passing through the image of any other vertex. In addition, we impose the following conditions on a drawing: (1) no three edges have an interior point in common, (2) if two edges share an interior point $p$, then they cross at $p$, and (3) any two edges of a drawing have only a finite number of crossings (common interior points). The crossing number $cr(G)$ of a graph $G$ is the minimum number of edge crossings in any drawing of $G$. Let $D$ be a drawing of the graph $G$, and we denote the number of crossings in $D$ by $cr(D)$. For more on the theory of crossing numbers, we refer the reader to [8]. The Cartesian product $G\times H$ of graphs $G$ and $H$ has the vertex set $V(G)\times V(H)$ and the edge set $E(G\times H)$ such that $\{(x_1, y_1), (x_2, y_2)\} \in E(G\times H)$ only if $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. The crossing number $cr(G\times H)$ can be determined using various methods [11].

The investigation of the crossing number of a graph is a classical but very difficult problem (for example, see [8]). In fact, computing the crossing number of a graph is NP-complete [9], and the exact values are known only for very restricted classes of graphs. The crossing numbers of the Cartesian products of graphs have been studied since 1973, when Harary et al. [12] conjectured that $cr(C_mQ_n) \geq (m-2)n$ for $m \geq 3$. This conjecture has been verified in [120–123] for $m \geq 4n$. Glebský and Salazar [10] also showed that the conjecture holds for $n \geq m+1$. The crossing number of the products of all 4-vertex graphs with paths and stars except $cr(K_3Q_P)$, which was earlier determined by Jendroľ and Ščerbová [13], who also obtained $cr(K_3Q_C)$ for $n \geq 1$. In their paper, they conjectured that $cr(S_mQ_P) = (n - 1)[\frac{m-2}{2}]$ for $m \geq 1$. For general $n$, the conjecture was recently confirmed by Bokal in [5]. Beineke and Ringeisen [3,4] determined the crossing numbers of the products of all 4-vertex graphs with cycles. Klešč [16–18] determined the crossing numbers of the products of all 5-vertex graphs with paths. In particular, he proved that $cr(K_5Q_P) = 6n$ for $n \geq 1$ in [16]. Zheng et al. [26] recently proved that $cr(K_5Q_P) = 15n + 3$ for $n \geq 1$ and

$$cr(K_nQ_n) = (n - 1)cr(K_{n+2} \setminus e) + 2cr(K_{m+1}).$$

In their paper, for $n \geq 1$, they conjecture that

$$cr(K_nQ_n) = \frac{1}{4} \begin{bmatrix} \frac{m+1}{2} & \frac{m-1}{2} & \frac{m-2}{2} & \frac{m+4}{2} & \frac{m-4}{2} \\ \frac{m+1}{2} & \frac{m-1}{2} & \frac{m-2}{2} & \frac{m+4}{2} & \frac{m-4}{2} \end{bmatrix},$$

for $n \geq 1$, $1 \leq m \leq 10$, which is consistent with Zheng’s conjecture for the crossing number of $K_{n}Q_{m}$.© 2018 IJRAR September 2018, Volume 5, Issue 3 www.ijrar.org (E-ISSN 2348-1269, P-ISSN 2349-5138)
In this contribution, we show that equality holds in (1.1) for \( m \geq 1 \) and conjecture that (1.2) holds for \( 1 \leq m \leq 10 \). The approach is seemingly new. To obtain the crossing number of \( K_m \), we construct a drawing of \( K_m \) from the drawing of \( K_m \) and obtain two lower bound expressions of \( cr(K_m) \) by the standard counting method used in [14,25]. To prove that equality holds in (1.1), we introduce the zip product operation that was used in [2,5,6,24] and prove a lemma about it with similar sufficient conditions to the ones in [5].

2. Some definitions and lemmas

**Definition 2.1.** For a graph \( G \), let \( A, B \subseteq E(G) \); then, for a drawing \( \varphi \) of \( G \), let

\[
\text{cr}(A, B) = \left| \varphi(a) \cap \varphi(b) \right|
\]

Additionally, let \( \text{cr}(A, A) = \text{cr}(A) \).

Informally, \( \text{cr}(A, B) \) denotes the number of crossings between every pair of edges where one edge is in \( A \) and the other in \( B \).

For three mutually disjoint subsets \( A, B, C \subseteq E(G) \), the identities

\[
\text{cr}(A \cup B) = \text{cr}(A) + \text{cr}(B) + \text{cr}(A, B)
\]

(2.1)

and

\[
\text{cr}(A, B \cup C) = \text{cr}(A, B) + \text{cr}(B, C)
\]

(2.2)

are noted.

Let \( G_i, i = 1, 2 \), be a graph with a vertex \( V_i \subseteq V(G_i) \) whose neighborhood \( N_i = N_{G_i}(V_i) \) has size \( d \). A zip function of graphs \( G_1, G_2 \) at vertices \( V_1 \) and \( V_2 \) is a bijection \( \sigma: N_i \rightarrow N_j \). The zip product \( G_1 \circ G_2 \) of graphs \( G_1 \) and \( G_2 \) according to \( \sigma \) is obtained from the disjoint union of \( G_1 \cup G_2 \) by adding the edges \( u \sigma(u), u N_{G_i} \) for \( i = 1 \). A drawing \( D_i \) of the graph \( G_i (i = 1, 2) \) defines (up to a circular permutation) a bijection \( \pi_i \) \( N_i \rightarrow 1, 2, \ldots, d \) that respects the edge rotation around \( V_i \) in \( D_i \). The zip function of drawings \( D_1 \) and \( D_2 \) at vertices \( V_1 \) and \( V_2 \) is \( \sigma = \sigma N \rightarrow N_2 \), \( \sigma = \pi_i \pi_j \). The zip product \( D_1 \circ D_2 \) of \( D_1 \) and \( D_2 \) according to \( \sigma \) is obtained from \( D_1 \) by adding a mirrored copy of \( D_2 \) that has \( V_2 \) incident with the infinite face disjointly into some face of \( D_1 \) incident with \( V_1 \) by removing vertices \( V_1 \) and \( V_2 \) and small disks around them from the drawings and then joining the edges according to the function \( \sigma \). For more detail, we refer the reader to [5,6]. For this construction, the following lemmas hold:

**Lemma 2.1 ([5]).** For \( i = 1, 2 \), let \( D_i \) be an optimal drawing of \( G_i \), let \( V_i \subseteq V(G_i) \) be a vertex of degree \( d \), and let \( \sigma \) be a zip function of \( D_i \) and \( D_j \) at \( V_i \) and \( V_j \). Then, \( \text{cr}(D_1 \circ D_2, G_1 \circ G_2) \leq \text{cr}(G_1) + \text{cr}(G_2) \).

Let \( S = K_{1,n} \) be a star graph with \( n \) vertices of degree 1 (called the leaves of the star) and one vertex of degree \( n \) (the center). Let \( G \) be a graph and \( S \subseteq V(G), |b| = n \). We say that \( S \) is \( k \)-star-connected in \( G \) if there exist \( k \) disjoint sets \( F_1, F_2, \ldots, F_{k} \subseteq E(G) \) such that either \( G - F \) is a subdivision of \( S \) with \( S \) being the leaves or \( G - F \) is a subdivision of \( S \) with all its leaves and the center belonging to \( S \).

**Lemma 2.2 ([5]).** Let \( G_1 \) and \( G_2 \) be \( 2 \)-star-connected graphs, \( V_i \subseteq V(G_i), \deg(V_i) = d \), and let the neighborhood \( N_i \) of \( V_i \) be \( 2 \)-star-connected in \( G_i - V_i, i = 1, 2 \). Then, \( \text{cr}(G_1 \circ G_2) \geq \text{cr}(G_1) + \text{cr}(G_2) \) for any bijection \( \sigma: N_i \rightarrow N_j \).

**Lemma 2.3 ([26]).** \( \text{cr}(K_m \setminus e) \leq 1 - \frac{m+4}{2} \frac{m-1}{2} \frac{m-2}{2} \frac{m-3}{2} \).

Some of the proofs in this paper are based on these results for the crossing numbers of complete graphs, more precisely as follows:

**Conjecture 2.1 ([11]).** \( \text{cr}(K_m) = 1 - \frac{m-2}{2} \frac{m-3}{2} \).

It has been proven by Guy [11] for \( m \leq 10 \) and by Pan and Richter [19] for \( m = 11, 12 \), respectively.
3. Lower bounds for \( \text{cr}(K_m \setminus e) \)

First, we give the following lower bound of the crossing number of \( K_m \setminus e \) in terms of the crossing numbers of \( K_{m-1} \) and \( K_m \).

\[
\text{cr}(K_m \setminus e) \geq \frac{2}{m} \text{cr}(K_{m-1}) + (m-2) \text{cr}(K_m) - \frac{m-3}{2}, \text{ for } m \geq 2.
\]

**Proof.** The graph \( K_m \setminus e \) has \( m - 2 \) vertices of degree \( m - 1 \), denoted by \( V_i, i = 1, 2, \ldots, m - 2 \), and two vertices of degree \( m - 2 \), denoted by \( x \) and \( y \), respectively. Let \( T^x \) and \( T^y \) denote the subgraphs induced by \( m - 2 \) edges incident with the vertex \( x \) and \( y \) respectively. It is easy to see that the subgraph induced by \( m - 2 \) vertices \( V_1, V_2, \ldots, V_{m-2} \) is isomorphic to the complete graph \( K_{m-2} \). Thus, we have

\[
K_m \setminus e = K_{m-2} \cup T^x \cup T^y.
\]

Let \( K^x_{m-1} = K_{m-2} \cup T^x \) and \( K^y_{m-1} = K_{m-2} \cup T^y \), respectively. It is easy to see that \( K^x_{m-1} \cong K_{m-1} \) and \( K^y_{m-1} \cong K_{m-1} \). Let \( D \) be an optimal drawing of \( K_m \setminus e \). Using (2.1) and (2.2), we have

\[
\text{cr}(K_m \setminus e) = \text{cr}(K^x_{m-1}) + \text{cr}(T^x, K_{m-2}) + \text{cr}(T^y, T^y);
\]

\[
\text{cr}(K_m \setminus e) = \text{cr}(K^y_{m-1}) + \text{cr}(T^y, K_{m-2}) + \text{cr}(T^x, T^x).
\]

Summing (3.1) and (3.2), we obtain

\[
2\text{cr}(K_m \setminus e) = \text{cr}(K^x_{m-1}) + \text{cr}(K^y_{m-1}) + \text{cr}(T^x \cup T^y, K_{m-2}) + 2\text{cr}(T^y, T^x).
\]

For \( i = 1, 2, \ldots, m-2 \), \( a_i \) denote the numbers of crossings on the path \( xV_iy \). We can see that in the plane \( R^2 \), there exists a circle neighborhood \( N(D(V_i), \varepsilon) \) where \( \varepsilon \) is a sufficiently small positive number such that for any other edge \( e' \) of \( K_m \setminus e \) incident with \( V_i \) no crossing appears on the segment \( D(e') \cap N(D(V_i), \varepsilon) \).

For \( 1 \leq i \leq m-2 \), we will obtain a drawing \( D' \) of \( K_m \) from \( D \). For this purpose, we draw a new edge from \( x \) to \( y \) (written as \( xy \)) first depart from vertex \( x \) near the edge \( xV_i \) then bypass vertex \( V_i \) in \( N(D(V_i), \varepsilon) \), and finally connect to \( y \) near to the edge \( yV_i \) (see Fig. 1, where the circuit \( C \) denotes the boundary of \( N(D(V_i), \varepsilon) \)). It is not difficult to see that the edge \( xy \) can cross the edge of \( K_m \setminus e \) exactly \( a_i + m \) times in \( D' \), where \( m \) denotes the minimal number of edges that lie on the same side of path \( xV_iy \). Clearly, \( m \leq \frac{m^2 - 2}{2} \).

\[
a_i = \text{cr}(K_m) - \text{cr}(K_{m-1}) - m.
\]

By implication,

\[
a_i = \text{cr}(K_m) - \text{cr}(K_{m-1}) - (m - 2)\text{cr}(K_{m-1}) - m.
\]

Additionally, it is easy to see that

\[
a_i = \text{cr}(K_{m-2}, T^x \cup T^y) + 2\text{cr}(T^y, T^x).
\]

By combining (3.3) and (3.6), we obtain

\[
2\text{cr}(K_m \setminus e) = \text{cr}(K^x_{m-1}) + \text{cr}(K^y_{m-1}) + \sum_{i=1}^{m-2} a_i.
\]
Again, from (3.5) and (3.7), we have
\[
mcr(K_n \setminus e) = \sum_{i=1}^{m-2} \frac{m-2}{m} \frac{m-2}{m} \crd(K_n) - m_i
\]
\[
\geq 2cr(K_{n-1}) + (m-2)cr(K_n) - (m-2) \frac{m-3}{2}
\]
as stated. \( \Box \)

**Theorem 3.2.** \( cr(K_n \setminus e) \geq \frac{m+2}{m} cr(K_{n-1}), \) for \( m \geq 3. \)

**Proof.** We will use the same notation as in Theorem 3.1. From (3.8), we take that
\[
mcr(K_n \setminus e) = \sum_{i=1}^{m-2} \frac{m-2}{m} \frac{m-2}{m} \crd(K_{n-1}) + \crd(K_{n-1}) + \crd(K_n) - m_i.
\]
Set \( \crd(K_{n-1}) = cr(K_{n-1}) + l_i, \) \( \crd(K_{n-1}) = cr(K_{n-1}) + l_2, \) where \( l_i \geq 0, l_2 \geq 0, \) \( m_i = cr(K_n) + c_i, \) and \( c_i \geq 0, \)
\( 1 \leq i \leq m - 2. \)

Thus, equality (3.9) can be written as
\[
mcr(K_n \setminus e) = 2cr(K_{n-1}) + l + l_2 + (m-2)cr(K_n) + c_i - m_i.
\]

Define the responsibility, \( r_\varphi(V), \) of a vertex \( V \) in a drawing \( \varphi \) as the total number of crossings on all edges incident to that vertex. Because each crossing is in the responsibility of 4 vertices, the total responsibility of all vertices \( r_\varphi(V) = 4cr(\varphi). \)

For more detail, we refer the reader to [11].

Thus, it is clear that
\[
r_\varphi(x) = cr(K_n \setminus e) - cr(K_{n-1})
\]
\[
= cr(K_n \setminus e) - cr(K_{n-1}) - l_2,
\]
and
\[
r_\varphi(y) = cr(K_n \setminus e) - cr(K_{n-1})
\]
\[
= cr(K_n \setminus e) - cr(K_{n-1}) - l_1.
\]

Similarly, for \( 1 \leq i \leq m - 2, \)
\[
r_\varphi(V_i) = cr(K_n) - cr(K_n \setminus V_i)
\]
\[
\leq cr(K_n) + c_i - cr(K_{n-1}).
\]

However, note also that
\[
r_\varphi(V_i) = r_\varphi(V_i) + m_i.
\]

Using (3.13) and (3.14), we obtain
\[
r_\varphi(V_i) \leq cr(K_n) - cr(K_{n-1}) + c_i - m_i.
\]

Therefore, it now follows from the definition of responsibility, together with (3.11), (3.12) and (3.15), that
\[
4cr(K_n \setminus e) = r_\varphi(x) + r_\varphi(y) + \sum_{i=1}^{m-2} \frac{m-2}{m} \frac{m-2}{m} r_\varphi(V_i)
\]
\[
\leq 2cr(K_n \setminus e) - 2cr(K_{n-1}) - (l_1 + l_2) + (m-2)cr(K_n) - (m-2)cr(K_{n-1}) + \frac{m-3}{m} \frac{m-3}{m} c_i - m_i.
\]

which yields
\[
2cr(K_n \setminus e) \leq (m-2)cr(K_n) - mcr(K_{n-1}) - (l_1 + l_2) + c_i - m_i.
\]

Therefore, by combining (3.10) and (3.16), we have
\[
(m-2)cr(K_n \setminus e) \geq (m + 2)cr(K_{n-1}) + 2(l_1 + l_2);\]
that is to say, we obtain the following inequality
\[ cr(K_n \setminus e) \geq \frac{m+2}{2} cr(K_{n-1}) + \frac{2(l_1 + l_2)}{m - 2} \]
\[ \geq \frac{m+2}{m - 2} cr(K_{n-1}) \]
as desired. \( \Box \)

**Theorem 3.3.** If Conjecture 2.1 is true for \( m = 2M - 1 \), where \( M \geq 2 \), then the equality holds in Theorem 3.2 for \( m = 2M \), i.e.,
\[ cr(K_{2M} \setminus e) = \frac{1}{4} (M^2 - 1)(M - 2)^2. \]

**Proof.** Lemma 2.3 clearly suffices to prove
\[ cr(K_{2M} \setminus e) \geq \frac{1}{4} (M^2 - 1)(M - 2)^2. \] (3.17)

If Conjecture 2.1 is true for \( m = 2M - 1 \), then
\[ cr(K_{2M-1}) = \frac{1}{4} \begin{array}{cccc} 2M - 1 & 2M - 1 & 2M - 2 & 2M - 2 \\ 2 & 2 & 2 & 2 \\ \end{array} \geq \frac{1}{4} (M - 1)(M - 2)^2. \]
Therefore, inequality (3.17) follows from Theorem 3.2, and the proof is completed. \( \Box \)

**Theorem 3.4.** \( cr(K_m \setminus e) = \frac{1}{4} + \frac{m^2}{m-2} \), for \( 1 \leq m \leq 12 \).

**Proof.** The cases for \( m \leq 5 \) are trivial. Because Conjecture 2.1 is true for \( m \leq 12 \), the theorem follows from Theorem 3.3 for \( m \leq 8, 10, 12 \). Assume now \( m \geq 12 \). Then, Lemma 2.3 shows that \( cr(K_{7} \setminus e) \geq 6 \). Moreover, applying Theorem 3.1, we have
\[ cr(K_7 \setminus e) \geq \frac{1}{4} [2cr(K_6) + 5(cr(K_7) - 2)] \]
\[ = \frac{41}{7}. \]
Additionally, note that \( cr(K_m \setminus e) \) is an integer, which implies that \( cr(K_7 \setminus e) \geq 6 \), and the theorem holds for \( m = 7 \). The proofs for \( m = 9, 11 \) are similar to the proof for \( m = 7 \), and the details are left to the reader. \( \Box \)

**Remark 3.1.** As an intuitive aid, in Table 1, we summarize the known crossing numbers for \( K_{m} \) and \( K_m \setminus e \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \leq 4 )</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( cr(K_m) )</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>18</td>
<td>36</td>
<td>60</td>
<td>100</td>
<td>150</td>
</tr>
<tr>
<td>( cr(K_m \setminus e) )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>15</td>
<td>30</td>
<td>54</td>
<td>90</td>
<td>140</td>
</tr>
</tbody>
</table>

4. Zip product

Let \( G \) be a graph and \( S \subseteq V(G) \), \( |S| = n \). We say that \( S \) is \( k \)-strict-star-connected in \( G \) if there exist \( k \) different sets \( F_1, F_2, \ldots, F_k \subseteq E(G) \) (possibly with common edges) such that either \( G[F_i] \) is a star \( S_k \) with \( S \) as the leaves or \( G[F_i] \) is a star \( S_{k-1} \) with all its leaves and the center belonging to \( S \). We call each \( F_i \) a star-set of \( S \).

**Lemma 4.1.** Let \( G_1 \) and \( G_2 \) be disjoint graphs, \( V_i \subseteq V(G_i) \), \( \deg(V_i) = d_i \), and let the neighborhood \( N_i \) of \( V_i \) be 4-strict-star-connected in \( G_i \). Then, \( cr(G_1 \circ \sigma G_2) \geq cr(G_1) + cr(G_2) \) for any bijection \( \sigma : N_1 \rightarrow N_2 \).

**Proof.** Let \( F = \{ u \sigma(u) | u \in N_{G_1}(V_1) \} \) be a set of edges joining \( G_1 \) and \( G_2 \). Then, \( G_1 \circ \sigma G_2 \) is a graph with \( E(F) \) and \( E(G_1 \circ \sigma G_2) \) different star-sets of \( N_i \). Let \( G_0 \) be the subgraph of \( G \) generated with the edges of \( E(F) \). We define the graph \( G_{ij} \) similarly. Clearly, \( G_{ij} \) is isomorphic to a subdivision of \( G_0 \), which implies that \( cr(G_{ij}) = cr(G_0) \).
Let $D$ be an optimal drawing of $G$. It is not difficult to show that

$$cr(G) \leq cr(E_1, E_1) + cr(E_1, F).$$

(4.1)

By summing (4.1) for $1 \leq j \leq 4$ and noting that any $e \in E_2$ is contained in at most two $F_2 \subseteq E_2$, we have

$$4cr(G) \leq 4cr(E_1, E_1) + 4cr(E_1, F) + \sum_{j=1}^{4} cr(E_1, F_2/j).$$

(4.2)

Similarly, we have

$$4cr(G) \leq 4cr(E_2, E_2) + 4cr(E_2, F) + \sum_{j=1}^{4} cr(E_2, F_1/j).$$

(4.3)

Combining inequalities (4.2) and (4.3), we have

$$4cr(G) + 4cr(G) \leq 4cr(E_1, E_1) + 4cr(E_2, E_2) + 4cr(E_1 \cup E_2, F) + 2cr(E_1, F) + 2cr(E_2, F) + \sum_{j=1}^{4} cr(E_1, F_2/j) \leq 4cr(G).$$

Thus, the proof is completed. Q

**Remark 4.1.** We can find that star-connected is different from strict-star-connected according to their definitions. Star-connected star-sets are pairwise disjoint, which is not required for strict-star-connected. However, it seems that requiring 4-strict-star-connected in Lemma 4.1 is too strong a condition. We propose the following question:

**Question 4.1.** Does $k$-strict-star-connected ($k \leq 3$) imply the claim of Lemma 4.1?

5. The crossing number of $K_n\hat{Q}_P$.

By $G + nK_1$, we denote a joint product of $G$, that is, the graph obtained from $G$ by adding $n$ vertices $V_i$ and the edges $V_iV_j$ for each $V \in V(G)$, $i = 0, 1, \ldots, n - 1$. For simpler labeling, let $G_0$ and $G_{0i}$ denote the graphs $G + K_1$ and $G + 2K_i$, respectively (see Fig. 2). We call a vertex $V \in V(G)$ a dominating vertex of $G$ if it is adjacent to all other vertices in $G$. By $P_n$, we denote the path of length $n$ whose vertices are $0, 1, \ldots, n$. With $G\hat{Q}P_n$, we denote the capped Cartesian product of $G$ and $P_n$, i.e., the graph obtained from $G\hat{Q}P_n$ by adding two new vertices $V_0$ and $V_n$ and connecting $V_0$ with all the vertices of $G\hat{Q}(0)$ and $V_n$ with all the vertices of $G\hat{Q}(n)$ in $G\hat{Q}P_n$.

**Lemma 5.1** ([5]). Let $G$ be a graph with a dominating vertex. Then, $cr(G\hat{Q}P_n) = (n + 1)cr(G + 2K_n)$ for $n \geq 0$.

**Lemma 5.2.** Let $G$ be a graph. Then, $cr(G\hat{Q}P_n) \leq (n + 1)cr(G + 2K_n)$ for $n \geq 0$. 
Theorem 5.1. Let $G$ be a vertex transitive graph. Then, for $n \geq 1$, 
\[ \text{cr}(\text{GQP}_n) \leq (n - 1)\text{cr}(G + 2K_1) + 2\text{cr}(G + K_1). \]

Proof. Let $D$ be an optimal drawing of $G \circ \sigma_0 G_0$. For $i = 0, 1$, let $\sigma_i$ be the circular labeling of the vertices of $V(G)$ around $V_i$ in $D$. The vertex $v$ has $\pi_i^{-1}$ as the circular labeling of its neighborhood in the mirror drawing $D'$. Set $D_0 = D$. For odd $n$, let $D_n = D_{n-1} \circ \sigma_0 D$ using a vertex with labeling $\pi_1$ and a vertex with labeling $\pi_1^{-1}$, and for even $n$, let $D_n = D_{n-1} \circ \sigma_0 D$ using a vertex with labeling $\pi_0$ and a vertex with labeling $\pi_0^{-1}$. It is clear that $D_n$ is a drawing of $G \circ \sigma_0 G_0$. However, the definition of a zip function implies that $\text{cr}(D_n) = (n + 1)\text{cr}(D) = (n + 1)\text{cr}(G + 2K_1)$. The lemma follows. Q

Theorem 5.2. Let $D$ be an optimal drawing of $G \circ \sigma_0 G_0$ and $G_0$ respectively. We will obtain a drawing $D_n$ of $G \circ \sigma_0 G_0$, from $D_{n-2}$ and $D$.

For $n = 1$, set $D_1 = D \circ \sigma_0 D$, where $\sigma_0$ is a zip function of $D$ and $D$ according to vertices $V_0$ and $V_0$. The definition of a zip function implies that $\text{cr}(D_1) = 2\text{cr}(D) = 2\text{cr}(G + K_1)$. However, it is clear that $D_1$ is a drawing of $G \circ \sigma_0 G_0$. Thus, the theorem follows for $n = 1$.

For $n \geq 2$, set $D_n = (D \circ \sigma_0 D_{n-2}) \circ \sigma_0 D$, where $\sigma_0$ is a zip function of $D$ and $D_{n-2}$ according to $V_0$ and $V_0$ and $\sigma_0$ is a zip function of $D \circ \sigma_0 D_{n-2}$ and $D$ according to $V_{n-2}$ and $V_0$. As $G$ is a vertex transitive graph, it is not difficult to show that $D_n$ is a drawing of $G \circ \sigma_0 G_0$. It follows from the definition of a zip function and Lemma 5.2 that 
\[ \text{cr}(G \circ \sigma_0 G_0) \leq \text{cr}(D_n) \]
\[ = \text{cr}(D) + \text{cr}(D_{n-2}) + \text{cr}(D) \]
\[ \leq 2\text{cr}(G + K_1) + (n - 1)\text{cr}(G + 2K_1) \]
as desired. Q

Theorem 5.3. Let $G$ be a graph with four dominating vertices. Then, for $n \geq 1$, 
\[ \text{cr}(G \circ \sigma_0 G_0) \geq (n - 1)\text{cr}(G + 2K_1) + 2\text{cr}(G + K_1). \]

Proof. For $n = 1$, it is not difficult to show that $G \circ \sigma_0 G_0 = G \circ \sigma_0 G_0$, where $\sigma_0 : V(G) \circ \sigma_0 G_0$ maps a vertex $v \in V_n$ to its counterpart in $V_0$. Note that $V_0 \subseteq V(G)$ have 4-strict-star-connected neighborhoods in $G_0$, and Lemma 4.1 implies that $\text{cr}(G \circ \sigma_0 G_0) \geq 2\text{cr}(G_0) = 2\text{cr}(G + K_1)$.

For $n \geq 2$, we have $G \circ \sigma_0 G_0 = (G \circ \sigma_0 (G \circ \sigma_0 G_0) \circ \sigma_0 G_0$, where $\sigma_0 : V(G) \circ \sigma_0 G_0$ maps a vertex $v \in V_n$ to its counterpart in $V_0$. We will obtain a drawing $G \circ \sigma_0 G_0$, from $G \circ \sigma_0 G_0$, and $G_0$. Thus, it follows from Lemmas 4.1 and 5.1 that 
\[ \text{cr}(G \circ \sigma_0 G_0) = \text{cr}(G \circ \sigma_0 (G \circ \sigma_0 G_0) \circ \sigma_0 G_0) \]
\[ \geq 2\text{cr}(G_0) + \text{cr}(G \circ \sigma_0 G_0) \]
\[ = 2\text{cr}(G + K_1) + (n - 1)\text{cr}(G + 2K_1) \]
as stated. Q

Corollary 5.1. For $m \geq 4$, $n \geq 1$, we have 
\[ \text{cr}(K_m \circ \sigma_0 G_0) = (n - 1)\text{cr}(K_m \circ \sigma_0 G_0) + 2\text{cr}(K_m + 1). \]

Proof. It is clear that $K_m + 2K_1 \cong K_{m+2} \circ \sigma_0 G_0$ and $K_m + K_1 \cong K_{m+1}$. This claim is an immediate consequence of Theorems 5.1 and 5.2. Q

Corollary 5.2. For $m \geq 1$ and $1 \leq m \leq 10$, we have 
\[ \text{cr}(K_m \circ \sigma_0 G_0) = \frac{1}{4} \left( m + 1 \right) \left( m - 1 \right) \left( m - 2 \right) \left( m + 4 \right) + \frac{m - 4}{2}. \]

Proof. The cases for $m \leq 3$ are trivial. Because Conjecture 2.1 is true for $m \leq 12$, the claim follows immediately from Theorem 5.4 and Corollary 5.1 for $m \geq 4$. Q
Corollary 5.3. If Conjecture 2.1 is true for \( m = 2M + 1 \), where \( M \geq 6 \), then, for \( n \geq 1 \),
\[
\text{cr}(K_{2M} Q P_2) = \frac{1}{4} M(M - 1)(nM + M + 2n - 2).
\]

Proof. This claim is an immediate consequence of Theorem 3.3 and Corollary 5.1. Q

Acknowledgments

The authors are indebted to two anonymous referees for their Suggestions, which improved the presentation and made it more readable. The work was supported by the National Natural Science Foundation of China (No. 11301169 & 11371133), Scientific Research Fund of Hunan Provincial Education Department (No. 12B026) and Hunan Provincial Natural Science Foundation of China (No. 13JJ4110).

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Authors:

1) K VENKATA NARAYANA
   Assoc.Professor
   Dept of HBS
   VSM College of Engineering-Ramachandrapuram
   (Affiliated to JNTU Kakinada,AndhraPradesh -533255)

2) R V TR MANIKYAMBA V ,MSc (Mathematics),PGDCA
   Assoc.Professor
   Dept of HBS
   VSM College of Engineering-Ramachandrapuram
   (Affiliated to JNTU Kakinada,AndhraPradesh -533255)

3) GRANDHI SURESH
   Asst.Professor
   Dept of HBS
   VSM College of Engineering-Ramachandrapuram
   (Affiliated to JNTU Kakinada,AndhraPradesh -533255)