Convergence of Fourier Series of Integral Function on the Interval – An Overview

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Abstract

This paper looks at a Fourier series as a way of representing a periodic function as a (possibly infinite) sum of sine and cosine functions and it is analogous to a Taylor series. First-order differentials $dx$, $dy$, … were incomparably smaller than, or infinitesimal with respect to, the finite quantities $x, y, …$, and, in general, that an analogous relation obtained between the $(n+1)^{th}$-order differentials $d^{n+1}x$ and the $n^{th}$-order differentials $dx^n$. He also assumed that the $n^{th}$ power $(dx)^n$ of a first-order differential was of the same order of magnitude as an $n^{th}$-order differential $dx^n$, in the sense that the quotient $d^n x/(dx)^n$ is a finite quantity.

A function $f(x)$ is said to have period $P$ if $f(x+P)=f(x)$ for all $x$. Let the function $f(x)$ has period $2\pi$. In this case, it is enough to consider behavior of the function on the interval $[-\pi,\pi]$.

1. Suppose that the function $f(x)$ with period $2\pi$ is absolutely integrable on $[-\pi,\pi]$ so that the following so-called Dirichlet integral is finite:

$$\pi \int_{-\pi}^{\pi} |f(x)| dx < \infty;$$

2. Suppose also that the function $f(x)$ is a single valued, piecewise continuous (must have a finite number of jump discontinuities), and piecewise monotonic (must have a finite number of maxima and minima).

If the conditions 1 and 2 are satisfied, the Fourier series for the function $f(x)$ exists and converges to the given function (see also the Convergence of Fourier Series page about convergence conditions.) At a discontinuity $x_0$, the Fourier Series converges to

$$\lim_{\varepsilon \to 0} \frac{1}{2} [f(x_0-\varepsilon)-f(x_0+\varepsilon)].$$

The Fourier series of the function $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\},$$

where the Fourier coefficients $a_0$, $a_n$, and $b_n$ are defined by the integrals

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Key words: Fourier series, periodic function, integrals, calculus, differential
Introduction

Leibniz's calculus gained a wide audience through the publication and by Guillaume de L'Hôpital (1661–1704), of the first expository book on the subject, the *Analyse des Infiniments Petits Pour L'Intelligence des Lignes Courbes*. This is based on two definitions:

1. Variable quantities are those that continually increase or decrease; and constant or standing quantities are those that continue the same while others vary.

2. The infinitely small part whereby a variable quantity is continually increased or decreased is called the differential of that quantity.

And two postulates:

1. Grant that two quantities, whose difference is an infinitely small quantity, may be taken (or used) indifferently for each other: or (what is the same thing) that a quantity, which is increased or decreased only by an infinitely small quantity, may be considered as remaining the same.

2. Grant that a curve line may be considered as the assemblage of an infinite number of infinitely small right lines: or (what is the same thing) as a polygon with an infinite number of sides, each of an infinitely small length, which determine the curvature of the line by the angles they make with each other.

Following Leibniz, L'Hôpital writes $dx$ for the differential of a variable quantity $x$. A typical application of these definitions and postulates is the determination of the differential of a product $xy$:

$$d(xy) = (x + dx)(y + dy) - xy = y \ dx + x \ dy + dx \ dy = y \ dx + x \ dy.$$ 

Here the last step is justified by Postulate I, since $dx \ dy$ is infinitely small in comparison to $y \ dx + x \ dy$.

Objective:

This paper intends to explore functions that are periodic/aperiodic, the *Fourier series* is replaced by the Fourier transform. For functions of two variables that are periodic in both variables.

Fourier series

The insistence that infinitesimals obey precisely the same algebraic rules as finite quantities forced Leibniz and the defenders of his differential calculus into treating infinitesimals, in the presence of finite quantities, as if they were zeros, so that, for example, $x + dx$ is treated as if it were the same as $x$. The trigonometric basis in the Fourier series is replaced by the spherical harmonics. The Fourier series, as well as its generalizations, is essential throughout the physical sciences. This was justified on the grounds that differentials are to be taken as variable, not fixed quantities, decreasing continually until reaching zero. Considered only in the “moment of their evanescence”, they were accordingly neither something nor absolute zeros.
Thus differentials (or infinitesimals) \(dx\) were ascribed variously the four following properties:

1. \(dx \approx 0\)
2. neither \(dx = 0\) nor \(dx \neq 0\)
3. \(dx^2 = 0\)
4. \(dx \to 0\)

where “≈” stands for “indistinguishable from”, and “→ 0” stands for “becomes vanishingly small”. Of these properties only the last, in which a differential is considered to be a variable quantity tending to 0, survived the 19th century refounding of the calculus in terms of the limit concept.

Fourier Series of Even and Odd Functions

The Fourier series expansion of an even function \(f(x)\) with the period of \(2\pi\) does not involve the terms with sines and has the form:

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,
\]

where the Fourier coefficients are given by the formulas

\[
a_0 = \frac{2\pi}{\pi} \int_0^\pi f(x) dx = \frac{2\pi}{\pi} \int_0^\pi f(x) \cos nx dx.
\]

Accordingly, the Fourier series expansion of an odd \(2\pi\)-periodic function \(f(x)\) consists of sine terms only and has the form:

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin nx,
\]

where the coefficients \(b_n\) are

\[
b_n = \frac{2\pi}{\pi} \int_0^\pi f(x) \sin nx dx.
\]

Below we consider expansions of \(2\pi\)-periodic functions into their Fourier series, assuming that these expansions exist and are convergent.

Gibbs Phenomenon

If there is a jump discontinuity, the partial sum of the Fourier series has oscillations near the jump, which might increase the maximum of the partial sum above the function itself. This phenomenon is called Gibbs phenomenon. The amplitude of the “overshoot” at any jump point of a piecewise smooth function is about 18% larger (as \(n \to \infty\)) than the jump in the original function.

Let the function \(f(x) = \pi - x^2\) be defined on the interval \([0,2\pi]\). Find the Fourier series expansion of the function on the given interval and calculate the approximate value of \(\pi\).

To prove that the Fourier series of the function \(f(x) = x^2\) converges uniformly to \(f(x)\) on the interval \([-\pi,\pi]\).

The Fourier series of the function \(f(x) = \pi - x^2\) defined on the interval \([0,2\pi]\) is given by the formula

\[
f(x) = \pi - x^2 = \sum_{n=1}^{\infty} \sin nx.
\]

Investigate behavior of the partial sums \(f_N(x)\) of the Fourier series.
The Fourier series of a periodic function $f(x)$ of period

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{2\pi k x}{T} + \sum_{k=1}^{\infty} b_k \sin \frac{2\pi k x}{T},$$

Fourier coefficients $a_k$ and $b_k$

$$a_k = \frac{2}{T} \int_{0}^{T} f(x) \cos \frac{2\pi k x}{T} \, dx, \quad b_k = \frac{2}{T} \int_{0}^{T} f(x) \sin \frac{2\pi k x}{T} \, dx.$$

The normalization factors in front of the coefficients come from the fact that the cosine and sine functions as defined are orthogonal but not orthonormal. The factor of $\frac{1}{2}$ multiplying $a_0$ therefore comes from the fact that the normalization for $a_0$ is different, since

$$a_0 = \frac{2}{T} \int_{0}^{T} f(x) \, dx$$

Illustration

Fourier series of the square wave, for which the function over one period is

$$f(x) = \begin{cases} 
1 & \text{if } 0 \leq x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} \leq x < 1.
\end{cases}$$

Note that near the jump discontinuities for the square wave, the finite truncations of the Fourier series tend to overshoot. This is a common aspect of Fourier series for any discontinuous periodic function.
The Heat Equation and Spherical Harmonics:

Fourier originally devised the use of Fourier series as a method of solving the heat equation

$$\frac{\partial T}{\partial t} - \alpha \nabla^2 T = 0,$$

where $T$ is temperature, $t$ is time, and $\alpha$ is some constant.

As can be shown from functional analysis, the set of eigenfunctions of the operator in one dimension are complete, meaning that any function can be represented by a linear combination of them. In one dimension, these eigenfunctions are exactly the sine and cosine functions. Since the heat equation prominently features the operator $2\nabla^2$, by representing functions via their Fourier series, Fourier was able to solve for the asymptotic temperature distribution in a material given an initial temperature distribution.

In a higher-dimensional equation using the Schrödinger equation for the hydrogen atom, it is more appropriate to use the higher-dimensional generalization of the Fourier series, the spherical harmonics.

Because of the least squares property, and because of the completeness of the Fourier basis, we obtain an elementary convergence result.
Divergence

Since Fourier series have such good convergence properties, many are often surprised by some of the negative results. For example, the Fourier series of a continuous $T$-periodic function need not converge pointwise. The uniform boundedness principle yields a simple non-constructive proof of this fact.

In 1922, Andrey Kolmogorov published an article titled Une série de Fourier-Lebesgue divergente presque partout in which he gave an example of a Lebesgue-integrable function whose Fourier series diverges almost everywhere. He later constructed an example of an integrable function whose Fourier series diverges everywhere.

The choice of eigenvectors of the DFT matrix has become important in recent years in order to define a discrete analogue of the fractional Fourier transform—the DFT matrix can be taken to fractional powers by exponentiating the eigenvalues (e.g., Rubio and Santhanam, 2005). For the continuous Fourier transform, the natural orthogonal eigenfunctions are the Hermite functions, so various discrete analogues of these have been employed as the eigenvectors of the DFT, such as the Kravchuk polynomials (Atakishiyev and Wolf, 1997). The "best" choice of eigenvectors to define a fractional discrete Fourier transform remains an open question.

**Discrete Fourier transform (DFT)**

These converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT), which is a complex-valued function of frequency. The interval at which the DTFT is sampled is the reciprocal of the duration of the input sequence. An inverse DFT is a Fourier series, using the DTFT samples as coefficients of complex sinusoids at the corresponding DTFT frequencies. It has the same sample-values as the original input sequence. The DFT is therefore said to be a frequency domain representation of the original input sequence. If the original sequence spans all the non-zero values of a function, its DTFT is continuous (and periodic), and the DFT provides discrete samples of one cycle. If the original sequence is one cycle of a periodic function, the DFT provides all the non-zero values of one DTFT cycle. Discrete-time Fourier transform (DTFT) indicates that a convolution of two sequences can be obtained as the inverse transform of the product of the individual transforms. An important simplification occurs when one of sequences is $N$-periodic

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x_n \cdot \left[ \cos \left( \frac{2\pi kn}{N} \right) - i \cdot \sin \left( \frac{2\pi kn}{N} \right) \right],$$

(Eq. 1)
Conclusion

The Fourier series as described above suffices to represent any periodic function. One can also say that this means the trigonometric functions are a complete set for representing functions on a compact interval, since any periodic function may be represented by the function over just one finite period.

For arbitrary functions over the entire real line which are not necessarily periodic, no Fourier series will be everywhere convergent. In this case, however, it is possible to represent a function by its Fourier transform. Given a function \( f(x) \), its Fourier transform is written

\[
\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx
\]

One can treat the formula like the same inner product that defines the coefficients of the Fourier series. Previously, the coefficients were numbers indexed by a discrete variable \( k \). Now, the variable \( k \) is continuous, and the function gives the value of the "coefficient" of the oscillating function \( e^{-2\pi i k x} \), which is one of an uncountable set of trigonometric functions. It is also possible to define the Fourier transform exactly analogous to the Fourier series, where a real trigonometric basis is used rather than a complex basis.
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