Faa di Bruno's Formula for Generalizing the Chain Rule to Higher Derivatives- An Analysis

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Abstract

This paper looks to present a proof of Fàà di Bruno's formula as the method of proof (based on the multiplicative structure of higher-order cotangent spaces) clarifies, in addition, the origin of the strange numerical coefficients and the orders of derivatives in the formula. Generalized Fàà di Bruno’s formula-an extension of the original Fait debunk formula to multivariate functions-is useful in different domains of analysis, statistics, and computational calculus; this for example, and for a discussion of the computational aspects, also see . Since, several different proofs of this formula have been given, notably. Although higher-order cotangent spaces are well known, they have not been used before to this end. Remark that the multiplicative structure of first-order cotangent spaces vanish identically. Hence, this structure is only relevant to higher order. Divided differences are often considered as the coefficients of the Newton interpolating polynomial. Assume that x0,x1,…,xn are distinct; then the divided differences of f are recursively.

As an application of the generalized Fab diBruno’s formula, we obtain a linear recurrence procedure to calculate higher-order derivatives of the inverse mapping. First, of all, let us introduce some notations. Multi-indices of length n are the elements a! = (Qlr * . a,,) in Nn. The order of Q: is ICXJ = cq + . . . + (Ye. We set CY! = crl!. . .a,,!, and O! = 1. Let us consider two open subsets U ⊂ Rn, V c Iw”, and two differentiable maps of class . As in A homogeneous space M is a space with a transitive group action by a Lie group. Because a transitive group action implies that there is only one group orbit, M is isomorphic to the quotient space G/H where H is the isotropy group Gx. The choice of x ∈ M does not affect the isomorphism type of G/Gx because all of the isotropy groups are conjugate. Many common spaces are homogeneous spaces, such as the hypersphere,

\[ S^n \sim O(n+1)/O(n) \]

A representation of a group G is a group action of G on a vector space V by invertible linear maps. For example, the group of two elements \( \mathbb{Z}_2 = \{0, 1\} \) has a representation \( \phi \) by \( \phi (0) v = v \) and \( \phi (1) v = -v \). A representation is a group homomorphism \( \phi : G \rightarrow GL(V) \).

Key words: invertible linear maps., group homomorphism , Faà di Bruno’s formula
Introduction

The problem of finding an explicit expression for the -th derivative of a composite function is an old one. Let be a composite function. There are several ways to represent the -th derivative of the composite function. One of the best-known ways is given by Faà di Bruno’s formula

where the sum runs over all partitions of the integer, denotes the number of parts of size, and denotes the number of parts of the partition considered. The first example discovered of a map from a higher-dimensional sphere to a lower-dimensional sphere which is not null-homotopic. Its discovery was a shock to the mathematical community, since it was believed at the time that all such maps were null-homotopic, by analogy with homology groups.

The Hopf map arises in many contexts, and can be generalized to a map . For any point in the sphere, its preimage is a circle in . There are several descriptions of the Hopf map, also called the Hopf fibration.

As a submanifold of , the 3-sphere is

is a complex line bundle on . In fact, the set of line bundles on the sphere forms a group under vector bundle tensor product, and the bundle generates all of them. That is, every line bundle on the sphere is for some .

The sphere is the Lie group of unit quaternions, and can be identified with the special unitary group , which is the simply connected double cover of . The Hopf bundle is the quotient map .

Faà di Bruno’s formula is an identity in mathematics generalizing the chain rule to higher derivatives, named in honor of Francesco Faà di Bruno (1825–1888). As usual, Faà di Bruno’s formula is also described in terms of the Bell polynomials:

where the exponential partial Bell polynomial is given by an explicit expression

As such, this dates back to [4]. Faà di Bruno’s formula has played an important role in combinatorial analysis. By the formula, Hsu found some strange identities. It has also been applied in many branches of mathematics such as in numerical analysis [6], [7] and statistics [8], [9]. Recently, Faà di Bruno’s formula has even been applied to the computation of Lamé function derivatives of arbitrary order and the singular behavior of -th angular derivatives of analytic functions in the unit disk in the complex plane and positive harmonic functions in the unit ball in [11]. Consequently, Faà di Bruno’s formula has been widely studied and generalized. For instance, some extensions of the formula in the case of multicomposite functions were considered in . Further results on the formula in several variables were also obtained by . It is worth noting that a divided difference version of Faà di Bruno’s formula has been established by using the chain rule of divided differences in the recent papers. Further generalization is given by Wang and Xu.
Objective:

This paper intends to explore and generalize the divided difference form of Faà di Bruno’s formula. Applying this to multicomposite functions, we obtain some extensions of Faà di Bruno’s formula which generalize the results.

Algebra homomorphisms

Given a commutative ring $R$, an $R$-algebra $H$ is a Hopf algebra if it has additional structure given by $R$-algebra homomorphisms

$$\Delta : H \rightarrow H \otimes_R H$$

(comultiplication) and

$$\epsilon : H \rightarrow R$$

(counit) and an $R$-module homomorphism

$$\lambda : H \rightarrow H$$

(antipode) that satisfy the properties

1. Coassociativity:

$$(\iota \otimes \Delta) \Delta = (\Delta \otimes I) \Delta : H \rightarrow H \otimes H \otimes H.$$  

(4)

2. Counitarity:

$$m (I \otimes \epsilon) \Delta = I = m (\epsilon \otimes I) \Delta.$$  

(5)

3. Antipode property:

$$m (I \otimes \lambda) \Delta = I \epsilon = m (\lambda \otimes I) \Delta,$$  

(6)

where $I$ is the identity map on $H$, $m : H \otimes H \rightarrow H$ is the multiplication in $H$, and $\iota : R \rightarrow H$ is the $R$-algebra structure map for $H$, also called the unit map.

Faà di Bruno’s formula can be cast in a framework that is a special case of a Hopf algebra (Figueroa and Gracia-Bondía 2005).

$$
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \otimes H \\
| \Delta \downarrow & & \downarrow I \otimes \Delta \\
H \otimes H & \xrightarrow{\Delta \otimes \iota} & H \otimes H \otimes H
\end{array}
$$
Coassociativity

means that the above diagram commutes, meaning if the arrows were reversed and $m$ were exchanged for $\Delta$, a diagram illustrating the associativity of the multiplication within $H$ would be obtained. And since $\iota$ embeds the ground ring $R$ into $H$, the counit maps $H$ into $R$. The counitary property is similarly the dual of that satisfied by $\epsilon$. With $\epsilon$ and $\Delta$, an algebra becomes a bialgebra, but it is the addition of the antipode that makes $H$ a Hopf algebra. The antipode should be thought of as an inverse on $H$ similar to that which exists within a group, and the antipode is an anti-homomorphism at the level of algebras and co-algebras, meaning that

\[
\lambda (h h') = \lambda (h') \lambda (h) \tag{7}
\]

\[
\Delta \lambda (h) = (\lambda \otimes \lambda) \tau \Delta, \tag{8}
\]

where $\tau (h \otimes h') = h' \otimes h$, which is called the switch map. Moreover, as with the inverse operation in a group, in many cases, the antipode is an involution.

The prototypical examples of Hopf algebras are group rings, where $G$ is a finite group and $H = R [G]$ is a Hopf algebra via

\[
\Delta (g) = g \otimes g \tag{9}
\]

\[
\epsilon (g) = 1_R \tag{10}
\]

\[
\lambda (g) = g^{-1} \tag{11}
\]

for $g \in G$ and extend by linearity to all of $R [G]$.

For general Hopf algebras, the comultiplication is given in Sweedler notation. That is, if $h \in H$ then

\[
\Delta (h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}, \tag{12}
\]

which allows by coassociativity

\[
(I \otimes \Delta) \Delta (h) = (\Delta \otimes I) \Delta (h) \tag{13}
\]

\[
= \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \in H \otimes H \otimes H \tag{14}
\]

to be unambiguously written.

\[
\begin{array}{c}
H \\
\xrightarrow{\Delta} \\
H \otimes H \\
\xrightarrow{\Delta} \\
H \otimes H
\end{array}
\]
Hopf algebras

Can be categorized into different types by dualizing the distinctions one makes between algebras. For example, if \( H \) is commutative, this is equivalent to saying that \( m : H \otimes H \rightarrow H \) satisfies the property that \( m \circ \tau = m \) where \( \tau \) is the switch map mentioned above. Likewise, a Hopf algebra is said to be cocommutative if \( \tau \circ \Delta = \Delta \), that is, if the above diagram commutes. Moreover, commutativity and cocommutativity are independent properties, and so Hopf algebras can be considered that satisfy one or the other, or both, or neither properties.

Additionally, just as the linear dual of an algebra is an algebra, the dual of a Hopf algebra \( H \) is also a Hopf algebra, where the algebra structure of \( H \) becomes the coalgebra structure of \( H^* \), and vice-versa, and the antipode for \( H \) translates into an antipode for \( H^* \) in a canonical fashion.

Faà di Bruno’s formula gives an explicit equation for the \( n \)th derivative of the composition \( f \circ g (t) \). If \( f (t) \) and \( g (t) \) are functions for which all necessary derivatives are defined, then

\[
D^n f \circ g (t) = \sum_{k_1! \cdots k_n!} \frac{n!}{k_1! \cdots k_n!} (D^k f) (g (t)) \left( \frac{D g (t)}{1!} \right)^{k_1} \cdots \left( \frac{D^n g (t)}{n!} \right)^{k_n},
\]

where \( k = k_1 + \ldots + k_n \) and the sum is over all partitions of \( n \), i.e., values of \( k_1, \ldots, k_n \) such that

\[ k_1 + 2 k_2 + \ldots + n k_n = n \]

(Roman 1980).

It can also be expressed in terms of Bell polynomial \( B_{n,k} (x) \) as

\[
D^n f \circ g (t) = \sum_{k=0}^n (D^k f) (g (t)) B_{n,k} (D g (t), D^2 g (t), \ldots)
\]

(M. Alekseyev, pers. comm., Nov. 3, 2006).

Faà di Bruno’s formula can be cast in a framework that is a special case of a Hopf algebra (Figueroa and Gracia-Bondía 2005).

The first few derivatives for symbolic \( f \) and \( g \) are given by

\[
\frac{d}{dt} f \circ g (t) = f' (g (t)) g' (t)
\]

\[
\frac{d^2}{dt^2} f \circ g (t) = [g' (t)]^2 f'' (g (t)) + f' (g (t)) g'' (t)
\]

\[
\frac{d^3}{dt^3} f \circ g (t) = 3 g' (t) f''' (g (t)) g'' (t) + [g' (t)]^3 f^{(3)} (g (t)) + f' (g (t)) g''' (t).
\]
Lie Algebra Representation

A representation of a Lie algebra $\mathfrak{g}$ is a linear transformation

$$\psi : \mathfrak{g} \rightarrow M (V),$$

where $M (V)$ is the set of all linear transformations of a vector space $V$. In particular, if $V = \mathbb{R}^n$, then $M (V)$ is the set of $n \times n$ square matrices. The map $\psi$ is required to be a map of Lie algebras so that

$$\psi ([A, B]) = \psi (A) \psi (B) - \psi (B) \psi (A)$$

for all $A, B \in \mathfrak{g}$. Note that the expression $AB$ only makes sense as a matrix product in a representation. For example, if $A$ and $B$ are antisymmetric matrices, then $AB - BA$ is skew-symmetric, but $AB$ may not be antisymmetric.

The possible irreducible representations of complex Lie algebras are determined by the classification of the semisimple Lie algebras. Any irreducible representation $V$ of a complex Lie algebra $\mathfrak{g}$ is the tensor product $V = V_0 \otimes L$, where $V_0$ is an irreducible representation of the quotient $\mathfrak{g}_{ss}/\text{Rad} (\mathfrak{g})$ of the algebra $\mathfrak{g}$ and its Lie algebra radical, and $L$ is a one-dimensional representation.

A Lie algebra may be associated with a Lie group, in which case it reflects the local structure of the Lie group. Whenever a Lie group $G$ has a group representation on $V$, its tangent space at the identity, which is a Lie algebra, has a Lie algebra representation on $V$ given by the differential at the identity. Conversely, if a connected Lie group $G$ corresponds to the Lie algebra $\mathfrak{g}$, and $\mathfrak{g}$ has a Lie algebra representation on $V$, then $G$ has a group representation on $V$ given by the matrix exponential.

Conclusion

The well-known formula of Faà di Bruno’s for higher derivatives of a composite function has played an important role in combinatorics. In this paper we generalize the divided difference form of Faà di Bruno’s formula and give an explicit formula for the $n$-th divided difference of a multicomposite function. More generally, we establish the relationships of the Bell polynomials with respect to multicomposite functions. Applying these to multicomposite functions, we obtain some extensions of Faà di Bruno’s formula.

Any representation $V$ of $G$ can be restricted to a representation of any subgroup $H$, in which case, it is denoted $\text{Res}_H^G$. More surprisingly, any representation $W$ on $H$ can be extended to a representation of $G$, on a larger vector space $V$, called the induced representation.

Representations have applications to many branches of mathematics, aside from applications to physics and chemistry. The name of the theory depends on the group $G$ and on the vector space $V$. Different approaches are required depending on whether $G$ is a finite group, an infinite discrete group, or a Lie group. Another important ingredient is the field of scalars for $V$. The vector space $V$ can be infinite dimensional such as a Hilbert space. Also, special kinds of representations may require that a vector space structure is preserved. For instance, a unitary representation is a group homomorphism $\phi : G \rightarrow U (V)$ into the group of unitary transformations which preserve a Hermitian inner product on $V$. 
In favorable situations, such as a finite group, an arbitrary representation will break up into irreducible representations, i.e., $V = \bigoplus \mathcal{V}_i$ where the $\mathcal{V}_i$ are irreducible. For many groups, the irreducible representations have been classified.

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