NUMERICAL METHOD FOR TIME FRACTIONAL BIOHEAT TRANSFER EQUATION AND APPLICATIONS

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Abstract

The aim of this paper is to develop an explicit finite difference scheme for one dimensional time fractional bioheat transfer equation. We also show that the scheme is stable and convergent conditionally. As an application of this scheme numerical solution for a skin-heating model is discussed with the help of Mathematica software.

Keywords: Finite difference scheme, Time fractional bioheat equation, Mathematica.

1. INTRODUCTION

In the present scenario fractional calculus plays an important role in the various fields of scientific and engineering problems. Fractional partial differential equations have many applications in engineering, physics, hydrology and finance [8]. Therefore, partial differential equations of fractional order have been successfully used for modelling relevant physical process studied in [2, 5, 6, 8, 10]. In this connection as an interest, we develop the finite difference scheme for one dimensional time fractional bioheat equation (TFBHE). To date, the numerical methods and analysis of the fractional order partial differential equations are limited; some numerical methods for solving the space or time fractional partial differential equations have been studied in [2, 5, 6, 8]. The thermal life phenomenon and temperature behaviour in living tissue is well studied in an elegant book by Chato [1]. It is necessary for modern clinical treatments and medicines such as Cancer hyperthermia, Cryopreservation, Cryosurgery and thermal disease diagnostics. The problems of bioheat transfer in human body have been studied by many researchers and contributed for designing clinical thermal treatment equipments, which are useful for accurate medical diagnosis. In the year 1948, Penne's established a celebrated model "Analysis of tissue and arterial temperature in the resting human forearm " which is devoted for the study of the problem of bioheat transfer in living tissue [7], the main ingredient is the classical Fourier law. The Penne's equation is the most widely used in the study of various models in heat transfer in living tissue [3, 4].

The original one dimensional Pennes bioheat transfer equation is [7]:

$$\rho C \frac{\partial \theta^*}{\partial t} + w_b c_b \theta^* = \frac{\partial}{\partial x}\left(k(x)\frac{\partial \theta^*}{\partial x}\right) + Q_r,\ 0 < x < L \tag{1.1}$$

ρ −the density of tissue;
c −specific heat of tissue;
k −thermal conductivity of tissue;
w_b −blood perfusion rate;
c_b −specific heat of blood;
Q_r −the volumetric heat due to spatial heating which is constant;
L −distance between skin surface and the body core:

Note that $\theta^* = T(x, t) - T_s$, is the elevated tissue temperature, where T(x, t) represents the temperature and T_s is the skins steady state temperature. We consider the one dimensional time fractional bioheat transfer equation:

$$\rho c \frac{\partial \theta^*}{\partial t} + w_b c_b \theta^* = \frac{\partial}{\partial x}\left(k(x)\frac{\partial \theta^*}{\partial x}\right) + Q_r,\ 0 < x < L,\ 0 < \alpha < 1 \qquad (1.2)$$

where variable coefficient $k(x) > 0$. If $\alpha = 1$ then above equation becomes Pennes bioheat transfer equation. The Caputo fractional derivative of order α is defined as

$$\frac{\partial^\alpha \theta}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial \theta(x,\xi)}{\partial \xi}\frac{d\xi}{(t-\xi)^\alpha} & ,\ 0 < \alpha < 1 \\ \frac{d\theta(x,t)}{dt},\ \alpha = 1 \end{cases}$$

We divide equation (1.2) by ρc, we get

$$\frac{\partial \theta^*}{\partial t} + \frac{w_b c_b}{\rho c}\left(\theta^* - \frac{Q_r}{w_b c_b}\right) = \frac{1}{\rho c}\frac{\partial}{\partial x}\left(k(x)\frac{\partial \theta^*}{\partial x}\right),\ 0 < x < L,\ 0 < \alpha < 1 \qquad (1.3)$$

Now, we have to assume the following basic assumptions:

$\theta = \left(\theta^* - \frac{Q_r}{w_b c_b}\right)$, $a = \frac{w_b c_b}{\rho c} > 0$, $b = \frac{1}{\rho c} > 0$

We arrived at the one dimensional time fractional bioheat transfer equation with initial and boundary conditions:

$$\frac{\partial \theta}{\partial t} + a\,\theta = \frac{\partial}{\partial x}\left(k(x)\frac{\partial \theta}{\partial x}\right),\ 0 < x < L,\ 0 < \alpha < 1 \qquad (1.4)$$

initial condition : $\theta(x, 0) = 0$ (1.5)

boundary conditions : $\theta(0, t) = \theta_0$; $\theta_x(L, t) = 0$; $t \geq 0$ (1.6)

The structure of the paper is as follows: In section 2, we develop time fractional order explicit finite difference scheme for skin heating model governed by bioheat transfer equation. The section 3, is devoted for stability analysis of the model and the question of convergence is proved in section 4. Finally, we obtain the numerical solution of skin heating model using Mathematica software and by the numerical example it is shown that the numerical results are in good agreement with our theoretical analysis in the last section.

We developed a fractional order explicit finite difference scheme for one dimensional time fractional bioheat transfer equation (TFBHE) in the next section.

2 AN EXPLICIT FINITE DIFFERENCE SCHEME

We first introduce the finite difference approximation to discritize the time fractional derivative. We define $t_k = k\Delta t$, k = 0,1, ..., N, $x_i = i\Delta x$, i = 0, 1, ..., M, where $\Delta t = T/N$ and $\Delta x = L/M$ are the time and space steps respectively. Let $\theta(x_i; t_k)$, i = 0, 1, ..., M,

k = 0, 1, ... , N be the exact solution of the TFBHE (1.4)-(1.6) at mesh point $(x_i; t_k)$. Let θ_i^k be the numerical approximation of the point $(i\Delta x; k\Delta t)$. In the differential equation (1.4), the time fractional derivative term is approximated by the following scheme:

$$\frac{\partial^\alpha \theta(x_i, t_{k+1})}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{\theta(x_i, t_{j+1}) - \theta(x_i, t_j)}{\Delta t}\int_{j\Delta t}^{(j+1)\Delta t}\frac{1}{(t_{k+1}-\xi)^\alpha}\partial\xi$$

This is simplified as:

$$\frac{\partial^\alpha \theta(x_i,t_{k+1})}{\partial t^\alpha} = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\left[\theta(x_i, t_{j+1}) - \theta(x_i, t_j)\right] + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j-1}^{k} d_j\left[\theta(x_i, t_{k-j+1}) - \theta(x_i, t_{k-j})\right]$$

Where $d_j = [(j+1)^{1-\alpha} - j^{1-\alpha}]$, j= 0,1, 2, …..,k

Using the time fractional approximation for TFBHE (1.4) − (1.6), we get

$$\theta_i^{k+1} - \theta_i^{k} + \sum_{j=1}^{k} d_j\left[\theta_i^{k-j+1} - \theta_i^{k-j}\right] + r_1\left(\theta_i^{k+1} + \theta_i^{k-j}\right) = r_2\left[k(x_i)\theta_{i-1}^k - (k(x_i) + k(x_{i+1}))\theta_i^k + k(x_{i+1})\,\theta_{i+1}^k\right]$$

Where $r_1 = \frac{(a\Delta t)^\alpha\Gamma(2-\alpha)}{2}$, $r_2 = b\frac{(\Delta t)^\alpha\Gamma(2-\alpha)}{(\Delta x)^2}$

After simplification,

$\theta_i^{k+1} = c_1\mathrm{k}(x_i)\theta_{i-1}^k + c_2[1\text{-}r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))]\,\theta_i^k + c_1\mathrm{k}(x_{i+1})\theta_{i+1}^k - c_2\sum_{j=1}^k d_j\left[\theta_i^{k-j+1} - \theta_i^{k-j}\right]$

where $c_1 = \frac{r_2}{1+r_1}$, $c_2 = \frac{1}{1+r_1}$ and $d_j = [(j+1)^{1-\alpha} - j^{1-\alpha}]$, j= 0,1, 2, …..,k

The initial condition $\theta(x_j, 0) = 0$ implies $\theta_i^0 = 0$, i = 1, 2 ,..., M, as well as the boundary conditions $\theta(0, t_k) = \theta_0$ implies $\theta_0^k = \theta_0$, $\theta_x(L, t_k) = 0$ implies $\theta_{M+1}^k = \theta_{M-1}^k$ for k = 0, 1, 2, ...,N.

Therefore, discretization of the problem (1.4) − (1.6) is:

$$\theta_i^{k+1} = c_1\mathrm{k}(x_i)\theta_{i-1}^k + c_2[1\text{-}r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))]\,\theta_i^k + c_1\mathrm{k}(x_{i+1})\theta_{i+1}^k - c_2\sum_{j=1}^k d_j\left[\theta_i^{k-j+1} - \theta_i^{k-j}\right] \tag{2.1}$$

initial condition $\theta_i^0 = 0$, i = 1, 2 ,..., M, (2.2)

boundary conditions $\theta_0^k = \theta_0$, $\theta_{M+1}^k = \theta_{M-1}^k$ for k = 0, 1, 2, ...,N. (2.3)

Where $r_1 = \frac{(a\Delta t)^\alpha\Gamma(2-\alpha)}{2}$, $r_2 = b\,\frac{(\Delta t)^\alpha\Gamma(2-\alpha)}{(\Delta x)^2}$ and $c_1 = \frac{r_2}{1+r_1}$, $c_2 = \frac{1}{1+r_1}$

An explicit finite difference scheme have a truncation error of the order $O[(\Delta t)^{1-\alpha}, (\Delta x)^2]$;

Putting k = 0 in equation (2.1) and boundary conditions, we get

$$\theta_i^1 = c_1\mathrm{k}(x_i)\theta_{i-1}^0 + c_2[1\text{-}r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))]\,\theta_i^0 + c_1\mathrm{k}(x_{i+1})\theta_{i+1}^0 \tag{2.4}$$

putting k = 1,2,… ,N-1 in equation (2.1), we get

$$\sum_{j=1}^k d_j\left[\theta_i^{k-j+1} - \theta_i^{k-j}\right] = d_1\theta_i^k + \sum_{j=1}^{k-1}\left(d_{j+1} - d_j\right)\theta_i^{k-j} - d_k\theta_i^0$$

Therefore, equation (2.1) can be written as

$$\theta_i^{k+1} = c_1\mathrm{k}(x_i)\theta_{i-1}^k + c_2[1\text{-}\,d_1 - r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))]\,\theta_i^k + c_1\mathrm{k}(x_{i+1})\theta_{i+1}^k + c_2\sum_{j=1}^{k-1}\left(d_j - d_{j+1}\right)\theta_i^{k-j} + c_2 d_k\theta_i^0 \tag{2.5}$$

Therefore the system of algebraic equations (2.1)−(2.3) can be written in the following form of the matrix equations form:

$$\theta^1 = \mathrm{A}\,\theta^0 + \mathrm{C} \tag{2.6}$$

$$\theta^{k+1} = (\mathrm{A+D}) + \mathrm{C} + c_2\sum_{j=1}^{k-1}\left(d_j - d_{j+1}\right)\theta_i^{k-j} + c_2 d_k\theta_i^0 \tag{2.7}$$

$$\theta^0 = 0 \tag{2.8}$$

where $A = (a_{ij})$ is a square matrix of coefficients and $C = (c_1 k(x_1)\,\theta^0, ..., 0)^T$ is matrix of order $M \times 1$ and $D = diagonal(-c_2 d_1, ..., -c_2 d_1)$. The coefficients for i = 1, 2, ..., M and j = 1, 2, ... M are

$$a_{ij} = \begin{cases} 0, & when\ j \geq i+1 \\ c_1\mathrm{k}(x_{i+1}), & when\ j = i+1 \\ c_2\left[1 - d_1 - r_1 - r_2\left(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1})\right)\right], & when\ j = i \\ c_1\mathrm{k}(x_i), & when\ j = i-1 \\ 0, & when\ j \leq i-2 \end{cases}$$

3 STABILITY

In this section we discuss, the stability of solution of the discrete TFBHE (2.1) − (2.3) we prove the following result.

Theorem 3.1 *The solution of the time fractional explicit finite difference scheme* (2.1) − (2.3) *for TFBHE* (1.4) − (1.6) *is conditionally stable.*

Proof: Consider the matrix equation form of the discrete time fractional finite difference scheme (2.1) − (2.3),

$\theta^1 = A\,\theta^0 + C$ (3.1)

$\theta^{k+1} = (A+D)\,\theta^k + C + c_2\sum_{j=1}^{k-1}(d_j - d_{j+1})\,\theta_i^{k-j} + c_2 d_k \theta_i^0$ (3.2)

$\theta^0 = 0$ (3.3)

For k = 0 from (3.1), we have

$$A = \begin{pmatrix} a_{11} & c_1k(x_2) & & \cdots & & 0 \\ c_1k(x_2) & a_{22} & c_1k(x_3) & & & 0 \\ \vdots & \vdots\ddots & \vdots\ddots & \vdots\ddots & & \vdots \\ 0 & \dots & c_1k(x_i) & a_{ii} & c_1k(x_{i+1}) & 0 \\ \vdots & & & & & \\ & & \vdots\ddots & \ddots & \vdots\ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \wedge & a_{MM} \end{pmatrix}$$

is a square matrix of order M and $a_{ii} = c_2[1 - r_1 - r_2(k(x_i) + k(x_{i+1}))]$, $i = 1,2,\dots, M$ and $\wedge = c_1(k(x_M) + k(x_{M+1}))$, $\theta^k = (\theta_1^k, \theta_2^k, \dots, \theta_M^k)^T$ and $C = (c_1k(x_1)\theta_0, 0, \dots, 0)^T$
In equation (3.1) each component of the last column vector C is constant and hence the propagation of the error depends on matrix A. From matrix A, we have the central diagonal element for each row $i = 1(1)M$ is $a_{ii} = c_2[1 - r_1 - r_2(k(x_i) + k(x_{i+1}))]$ and sum of off diagonal elements for each row $i = 1(1)M$ is $c_1(k(x_i) + k(x_{i+1}))$.

Let λ be an eigenvalue of the matrix A to the linear system of equation (3.1). According to Greschgorin's theorem [4] shows that the each eigenvalue of matrix A lie in union of the circles centred at a_{ii} with radius $\sum_{j=1, j\neq i}^{M-1}|a_{ij}|$.

$|\lambda - a_{ii}| < \sum_{j=1, j\neq i}^{M-1}|a_{ij}|$ (3.4)

$|\lambda - c_2[1 - r_1 - r_2(k(x_i) + k(x_{i+1}))]| < c_1(k(x_i) + k(x_{i+1}))$ (3.5)

substituting c_1 and c_2 in equation (3.5) and solving it gives,

$$-\frac{r_2}{1+r_1}(k(x_i) + k(x_{i+1})) < \lambda - \frac{1}{1+r_1}[1 - r_1 - r_2(k(x_i) + k(x_{i+1}))] < \frac{r_2}{1+r_1}(k(x_i) + k(x_{i+1}))$$

on simplification which gives,

$$\frac{1-r_1}{1+r_1} - \frac{2r_2}{1+r_1}(k(x_i) + k(x_{i+1})) < \lambda < \frac{1-r_1}{1+r_1} < 1$$

For stability, we have $-1 < \lambda < 1$.

The equations will be stable when, $-1 < \frac{1-r_1}{1+r_1} - \frac{2r_2}{1+r_1}(k(x_i) + k(x_{i+1}))$

which implies, $r_2 < \frac{1}{(k(x_i)+k(x_{i+1}))} < \frac{1}{(k(x_M)+k(x_{M+1}))} \leq \frac{1}{2}$ for $i = 1, 2, \dots, M+1$

For k = 1,2,...,N-1, consider the equation (3.2),

$\theta^{k+1} = B\theta^k + C + c_2\sum_{j=1}^{k-1}(d_j - d_{j+1})\,\theta_i^{k-j} + c_2 d_k \theta_i^0$ (3.6)

where B = A + D. Here each component of the vectors C, θ^0 and $\sum_{j=1}^{k-1}(d_j - d_{j+1})\,\theta_i^{k-j}$ are constant and hence propagation of error depends on matrix B, where

$$B = \begin{pmatrix} a_{11} & c_1k(x_2) & & \cdots & & 0 \\ c_1k(x_2) & a_{22} & c_1k(x_3) & & & 0 \\ \vdots & \vdots\ddots & \vdots\ddots & \vdots\ddots & & \vdots \\ 0 & \dots & c_1k(x_i) & a_{ii} & c_1k(x_{i+1}) & 0 \\ \vdots & & & & & \\ & & \vdots\ddots & \ddots & \vdots\ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \wedge & a_{MM} \end{pmatrix}$$

is a square matrix of order M and $a_{ii} = c_2[1 - r_1 - r_2(k(x_i) + k(x_{i+1}))]$, $i = 1,2,\ldots$, M and $\wedge = c_1(\mathrm{k}(x_M) + \mathrm{k}(x_{M+1}))$. Again by using equation (3.4) we get,

$$-\frac{r_2}{1+r_1}\big(k(x_i)+k(x_{i+1})\big) < \lambda - \frac{1}{1+r_1}[1-d_1-r_1-r_2\big(k(x_i)+k(x_{i+1})\big)] < \frac{r_2}{1+r_1}\big(k(x_i)+k(x_{i+1})\big)$$

For stability we consider,

$$-1 < \frac{1}{1+r_1}[1-d_1-r_1-r_2\big(k(x_i)+k(x_{i+1})\big)] - \frac{r_2}{1+r_1}\big(k(x_i)+k(x_{i+1})\big)$$

$$\therefore r_2 < \frac{1}{(k(x_i)+k(x_{i+1}))} < \frac{1}{(k(x_M)+k(x_{M+1}))} \le \frac{1}{2} \text{ for } i = 1, 2, \ldots, M+1$$

Hence the proof.

4. CONVERGENCE

In this section we discuss the question of convergence.

Theorem 4.1 *The finite difference scheme* (2.1) − (2.3) *for TFBHE* (1.4) − (1.6) *is convergent.*

Proof: Let Ω be the region $0 < x < L$, $0 < t < T$: Take $(x_i, t_k) = (i\Delta x, k\Delta t)$ *for* $i = 0,1,\ldots, M$ *and* $k = 0, 1, \ldots, N$ *with* $M\Delta x = L$; $N\Delta t = T$: We introduce the vector $\theta^{**n} = [\theta(x_0, t_k),\ldots,\theta(x_i, t_k),\ldots,\theta(x_M, t_k)]^T$ satisfying the finite difference scheme (2.1) − (2.3). We get,

$$\theta^{**k+1} = c_1\mathrm{k}(x_i)\theta_{i-1}^{**k} + c_2[1 - r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))]\,\theta_i^{**k} + c_1\mathrm{k}(x_{i+1})\theta_{i+1}^{**k}$$

$$-c_2\sum_{j=1}^{k} d_j\big[\theta_i^{**k-j+1} - \theta_i^{**k-j}\big] + \tau^k \quad (4.1)$$

where τ^k is the vector of the truncation errors at level t_k,

$\theta^{**0} = 0$, $\theta^{**k} = \theta_0$, $\theta_{M+1}^{**k} = \theta_{M-1}^{**k}$, Now, subtract (2.1) from (4.1), we get,

$$(\theta^{**k+1} - \theta_i^{k+1}) = c_1\mathrm{k}(x_i)\,(\theta_{i-1}^{**k} - \theta_{i-1}^{k}) + c_2[1 - r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))]\,(\theta_i^{**k} - \theta_i^{k}) + c_1\mathrm{k}(x_{i+1})\,(\theta_{i+1}^{**k} - \theta_{i+1}^{k}) - c_2\sum_{j=1}^{k} d_j\big[\theta_i^{**k-j+1} - \theta_i^{k+1-j}\big] + c_2\sum_{j=1}^{k} d_j\big[\theta_i^{**k-j} - \theta_i^{k-j}\big] + \tau^k \quad (4.2)$$

We put $E_i^k = \theta_i^{**k} - \theta_i^k$, in equation (4.2), we get

$$E^{n+1} = A\,E^n + \tau^n \quad (4.3)$$

Clearly, E^n satisfies (2.1)-(2.3), we have

$$E_i^{k+1} = c_1\mathrm{k}(x_i)E_{i-1}^k + c_2[1 - r_1 - r_2\big(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1})\big)]\,E_i^k + c_1\mathrm{k}(x_{i+1})E_{i+1}^k - c_2\sum_{j=1}^{k} d_jE_i^{k+1-j}$$

$$+\sum_{j=1}^{k} d_jE_i^{k-j} + \tau^k \quad (4.4)$$

where $E_i^0 = 0$, $E_0^k = 0$, $E_{M+1}^k = 0$.

$$E_i^{k+1} = c_1 \mathrm{k}(x_i) E_{i-1}^k + c_2[1 - r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))] E_i^k + c_1 \mathrm{k}(x_{i+1}) E_{i+1}^k - c_2 \sum_{j=1}^k d_j (E_i^{k+1-j} - E_i^{k-j}) + \tau^k \quad (4.5)$$

Let $E^k = (E_1^k, E_2^k, \ldots, E_M^k)^{\mathrm{T}}$, $\|E^k\| = \max_{1\le i\le M} |E_i^k|$ and d = $\max_{1\le j\le k} |d_j|$

Therefore, from equation(4.4), we get

$$\|E^{k+1}\| \le \left| c_1 \mathrm{k}(x_i) + c_2[1 - r_1 - r_2(\mathrm{k}(x_i) + \mathrm{k}(x_{i+1}))] + c_1 \mathrm{k}(x_{i+1}) - c_2 \sum_{j=1}^k d \right| \|E^k\| + \max_{1\le k\le N} \tau^N \quad (4.6)$$

Substituting c_1 and c_2 in equation (4.6), we get

$$\|E^{k+1}\| \le \frac{1-r_1+d}{1+r_1} \|E^k\| + \max_{1\le k\le N} \|\tau^N\| \quad (4.7)$$

$\|E^0\| = 0$, implies $\|E^k\| = 0$

Hence, $\|E^{k+1}\| \le \max_{1\le k\le N} \|\tau^N\|$ (4.8)

Since, $\lim_{(\Delta x,\Delta t)\to(0,0)} \|\tau^N\| = 0$, implies that, $\|E^{k+1}\| \to 0$ uniformly in Ω as $(\Delta x,\ \Delta t) \to (0,0)$.
The proof is completed.

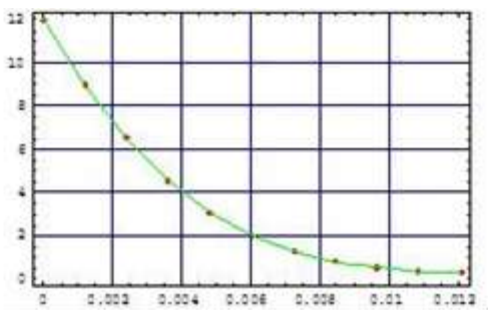
5. NUMERICAL SOLUTIONS

We obtained the numerical solution of one dimensional bioheat equation by an time fractional explicit finite difference scheme developed in equations (2.1) - (2.3). This is a one dimensional model of skin structure with a thickness of 0.01208m. In our test problem the values of physical properties are chosen, which are given as follows:
ρ = 1000 Kg/m3, c (= c_b) = 4200J/kg/^{0}C, w_b = 0.5 Kg/m3, L = 0.01208m. which are the same as in [3], only the thermal conductivity depends on x: The temperature is set to θ_0 = 12^{o}C to increase step of the skin surface. Our aim is to verify the stability of a new developed time fractional order explicit finite difference scheme. Here we consider the homogeneous tissue with the thermal conductivity of tissue being a function of x whose depth is measured in meter . For our test problem we perform the tests on three meshes 500, 750 and 1000 respectively with the time increments being Δt = 0:005s.

Test Problem :

We choose k(x) = 0.7(1 + 3x); which is a linear function of x and time Δt = 0:005s. Figure 5.1 shows the temperature profiles along the x direction at 150s,
Figure 5.2 represents the temperature elevations in the skin at x = 0:002416m: Using Mathematica software we obtain the simulations of heat transfer in skin by TFEFDS for three different mesh sizes.
Blue Meshes : 10 × 50 = 500; Red Meshes : 10 ×75 = 750; Green Meshes : 10 × 100 = 1000

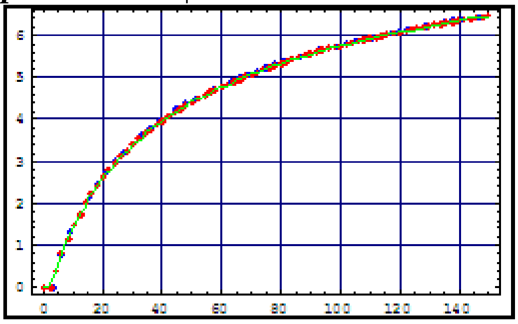
Temperature ^oC ↑



$\rightarrow$*distance* $x(m)$

*Fig:*5*:*1:Temperature profiles along x direction at $t = 150s$;[*with* $k(x) = 0.7(1 + 3x)$*;* $\Delta t = 0.005s$]

Temperature $\theta^{o}C\uparrow$



$\rightarrow$ ***Time*** $t(s)$

*Fig:*5.2 :Temperature elevations in skin at $x = 0.002416m$;[*with* $k(x) = 0.7(1 + 3x)$;$\Delta t = 0.005s$]

CONCLUSION:

(i) The numerical solutions of our test problem is independent of the mesh size.
(ii) It follows that our time fractional order explicit finite difference scheme is stable numerically also.
(iii) It is very difficult to handle the fractional order derivative problems and also obtain the numerical solution of the time fractional order explicit finite difference scheme.
(iv) To obtain the numerical solution of time fractional bioheat transfer equation by the time fractional order explicit finite difference scheme, CPU requires more time because it involves large matrices

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