Connected and Independent Geodetic Certified Domination Number of a graph

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ABSTRACT

Let G = (V, E) be any connected graph. A geodetic certified dominating set of G is a subset S of vertices of G which is both geodetic and certified dominating set of G. A connected geodetic certified dominating set S of G is a geodetic certified dominating set such that the subgraph (S) induced by S is connected. An independent geodetic certified dominating set S of G is a geodetic certified dominating set such that the subgraph (S) induced by S is independent. The minimum cardinality of a connected geodetic certified dominating set of G is the connected geodetic certified domination number and is denoted by \( c\gamma_{gc}(G) \). The minimum cardinality of an independent geodetic certified dominating set of G is the independent geodetic certified domination number and is denoted by \( c\gamma_{gi}(G) \). In this paper, connected and independent geodetic certified domination number of graphs are introduced and found for some standard graphs.

Key words: geodetic number geodetic certified domination number, connected geodetic certified domination number, independent geodetic certified domination number.

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1. Introduction

Throughout this paper G = (V, E) denotes a finite undirected simple graph with vertex set V and edge set E. The order \(|V|\) and size \(|E|\) of G denoted by p and q, respectively. For graph theoretic terminology we refer to [7]. A vertex v of G is said to be an extreme vertex if the subgraph induced by its neighbourhood is complete. The set of all extreme vertices is denoted by Ext(G). A vertex which is adjacent to an end vertex or pendent vertex is said to be support vertex. An x – y path of length d(x, y) is called geodesic. The closed interval \([x, y]\) consists of x, y and all vertices lying on some x – y geodesic of G, and for a non – empty set \( S \subseteq V(G) \), \( I[S] = \bigcup_{x,y \in S} [x,y] \). The concept of geodetic number of a graph was introduced by G. Chatrand and P. Zhang in [2]. A set \( S \subseteq V(G) \) of G is called a geodetic set of G if \( I[S] = V(G) \). The geodetic number of G, denoted by g(G), is the minimum cardinality of a geodetic set of G. A set \( S \subseteq V(G) \) is called a dominating set of G if for every vertex in \( V - S \) is adjacent to at least one vertex in S. A subset \( S \subseteq V(G) \) is called a certified dominating set of G if S is a dominating set of G and every vertex in S has either zero or at least two
neighbours in $V - S$. The certified domination number of $G$, denoted by $\gamma_{cer}(G)$, is the minimum cardinality of a certified dominating set of $G$ [3].

The concept of geodetic certified domination number was introduced by Dr. S. Joseph Robin and Mrs. L. Mary Vasanthi [6]. For a connected graph $G$, a set of vertices $S$ in $G$ is called a geodetic certified dominating set of $G$ if $S$ is both geodetic and certified dominating set. The minimum cardinality of a geodetic certified dominating set of $G$ is the geodetic certified domination number and is denoted by $c\gamma_g(G)$.

**Theorem 1.** [6] Every extreme vertex of $G$ belongs to every geodetic certified dominating set of $G$.

**Theorem 1.** [6] Each support vertex of $G$ belongs to every geodetic certified dominating set of $G$.

**Theorem 1.** [6] Let $G$ be a connected graph with cut vertices and let $S$ be a geodetic certified dominating set $G$. If $v$ is a cut vertex of $G$, then every component of $G - v$ contains an element of $S$.

### 2. Connected Geodetic Certified Domination Number.

**Definition 2.** Let $G$ be a connected graph with at least two vertices. A connected geodetic certified dominating set $S$ of $G$ is a geodetic certified dominating set such that the subgraph $\langle S \rangle$ induced by $S$ is connected. The minimum cardinality of a connected geodetic certified dominating set of $G$ is called the connected geodetic certified domination number of $G$. It is denoted by $c\gamma_{gc}(G)$. A geodetic certified dominating set of cardinality $c\gamma_{gc}(G)$ is called a $c\gamma_g$-set of $G$.

**Example 2.** Consider the graph $G$ in Figure 1.

![Figure 1](image_url)

Here $S = \{v_1, v_2, v_7\}$ is the unique minimum geodetic certified dominating set of $G$ and so $c\gamma_g(G) = 3$. But the induced subgraph $\langle S \rangle$ is not connected. So that $S$ is not a connected geodetic certified dominating set of $G$. Since $S_1 = \{v_1, v_2, v_3, v_7\}$ is a minimum connected geodetic certified dominating set of $G$, we have $c\gamma_{gc}(G) = 4$. Thus the geodetic certified domination number and the connected geodetic certified domination number are different.
Remark 2.3. In Figure 1, $S_1 = \{v_1, v_2, v_3, v_7\}, S_2 = \{v_1, v_2, v_4, v_7\}, S_3 = \{v_1, v_2, v_5, v_7\}$ and $S_4 = \{v_1, v_2, v_6, v_7\}$ are four different $c_{\gamma gc}$– set of $G$. Thus there can be more than one $c_{\gamma gc}$ – set for a connected graph.

**Theorem 2.4.** Every extreme vertex of a connected graph $G$ belong to every connected geodetic certified dominating set of $G$.

**Proof.** Since every connected geodetic certified dominating set is also a geodetic certified dominating set, the result follows from Theorem 1.1.

**Corollary 2.5.** For any complete graph $K_p (p \geq 2)$, $c_{\gamma gc}(K_p) = p$.

**Proof.** Since every vertex in the complete graph $K_p$ are extreme, the result follows from Theorem 2.4.

**Theorem 2.6.** For any connected graph $G$ of order $p \geq 2$, $2 \leq g(G) \leq c_{\gamma g}(G) \leq c_{\gamma gc}(G) \leq p$.

**Proof.** Any geodetic set needs at least two vertices and so $g(G) \geq 2$. Every geodetic certified dominating set is a geodetic set and it follows that $g(G) \leq c_{\gamma g}(G)$. Since every connected geodetic certified dominating set is also a geodetic certified dominating set, we have $c_{\gamma g}(G) \leq c_{\gamma gc}(G)$. Moreover, the set of all vertices of $G$ forms a connected geodetic certified dominating set of $G$, we have $c_{\gamma gc}(G) \leq p$. Hence, $2 \leq g(G) \leq c_{\gamma g}(G) \leq c_{\gamma gc}(G) \leq p$.

**Remark 2.7.** For the complete graph $K_p (p \geq 2)$, $g(K_p) = c_{\gamma g}(K_p) = c_{\gamma gc}(K_p) = p$. Also, all the inequalities in Theorem 2.6 can be strict. For the graph $G$ given in Figure 2, $S = \{v_1, v_9\}$ is the unique minimum geodetic set of $G$ and so $g(G) = 2$. The set $S_1 = \{v_1, v_5, v_9\}$ is the unique minimum geodetic certified dominating set of $G$ and so $c_{\gamma g}(G) = 3$. Also, $S_2 = \{v_1, v_2, v_5, v_6, v_9\}$ is a minimum connected geodetic certified dominating set of $G$ and so $c_{\gamma gc}(G) = 5$. Thus, $g(G) < c_{\gamma g}(G) < c_{\gamma gc}(G) < p$. 

![Figure 2](image-url)
**Corollary 2.8.** Let G be any connected graph. If $c_{\gamma gc}(G) = 2$, then $g(G) = c_{\gamma g}(G) = 2$.

**Corollary 2.9.** Let G be any connected graph of order p. If $g(G) = p$, then $c_{\gamma g}(G) = c_{\gamma gc}(G) = p$.

**Theorem 2.10.** Let G be a connected graph with cut – vertices and let S be a connected geodetic certified dominating set of G. If v is a cut vertex of G, then every component of $G - v$ contains an element of S.

**Proof.** Let v be a cut vertex of a connected graph G and S be a connected geodetic certified dominating set of G. Since every connected geodetic certified dominating set is also a geodetic certified dominating set, by Theorem 1.3, every components of $G - v$ contains at least one element of S.

**Theorem 2.11.** Every cut vertex of a connected graph G belong to every connected geodetic certified dominating set of G.

**Proof.** Let S be a connected geodetic certified dominating set of G and let v be a cut vertex of G. Let $G_1, G_2, \ldots, G_n (n \geq 2)$ be the components of $G - v$. By Theorem 2.10, S contains at least one vertex from each $G_i (1 \leq i \leq n)$. Since $\langle S \rangle$ is connected, we conclude that $v \in S$.

**Corollary 2.12.** For any tree T of order $p \geq 2$, $c_{\gamma gc}(T) = p$.

**Theorem 2.13.** Each support vertex of a connected graph G belong to every connected geodetic certified dominating set of G.

**Proof.** Since every connected geodetic certified dominating set of G is a geodetic certified dominating set, the result follows from Theorem 1.2.

**Theorem 2.14.** Let G be a connected graph. If every vertex of G is either an extreme vertex or support vertex or a cut – vertex of G. Then $c_{\gamma gc}(G) = p$.

**Proof.** This follows from Theorem 2.4, Theorem 2.11 and Theorem 2.13.

**Remark 2.15.** The converse of Theorem 2.14 need not be true. For the graph G in Figure 3, $S = V(G)$ is the only minimum connected geodetic certified dominating set of G and so $c_{\gamma gc}(G) = p$. 


Theorem 2.16. For any connected graph $G$, $c_{\gamma_{gc}}(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then by corollary 2.5 $c_{\gamma_{gc}}(G) = 2$. Conversely, assume $c_{\gamma_{gc}}(G) = 2$. Let $S = \{x, y\}$ be a minimum connected geodetic certified dominating set of $G$. Since $(S)$ is connected that $u$ and $v$ are adjacent. If $G \neq K_2$, then there exists a vertex $z$ different from $x$ and $y$. Since $xy$ is an edge, that $z$ does not lie on the $x - y$ geodesic and so that $S$ is not a connected geodetic certified dominating set of $G$, which is a contradiction. Hence $G = K_2$.

Theorem 2.17. For any pair $k, p$ of integer with $4 \leq k \leq p$ and $k \neq p - 1$, there exists a connected graph $G$ of order $p$ with $c_{\gamma_{gc}}(G) = k$.

Proof. If $k = p$, then take $G$ as a tree of order $p$. By corollary 2.12 $c_{\gamma_{gc}}(G) = p$. Now we consider the case for $4 \leq k < p$ and $k \neq p - 1$. Let $P_k: u_1, u_2, \ldots, u_{k-1}$ be a path on $(k - 1)$ vertices. Add new vertices $v_1, v_2, \ldots, v_{p-k+1}$ and join each $v_i$ ($1 \leq i \leq p - k + 1$) with $u_1$ and $u_3$, thereby obtaining the graph $G$ as given in Figure 4.
Clearly G is a connected graph of order p and S = \{u_3, u_4, \ldots, u_{k-1}, u_k\} be the set of extreme and cut vertices of G. Therefore, |S| = k – 2 and by Theorem 2.4, 2.11, \(\gamma_{gc}(G) \geq k - 2\). It is easily verified that S ∪ \{u_1, u_2\} is a minimum connected geodetic certified dominating set of G and so \(c_{\gamma_{gc}}(G) = k - 2 + 2 = k\).

3. Independent Geodetic Certified Domination Number

**Definition 3.1.** A geodetic certified dominating set S of G is said to be an independent geodetic certified dominating set of G if the subgraph induced by S is independent. The minimum cardinality among all independent geodetic certified dominating set of G is called the independent geodetic certified domination number of G. It is denoted by \(c_{\gamma_{gi}}(G)\). An independent geodetic certified dominating set of cardinality \(c_{\gamma_{gi}}(G)\) is called \(c_{\gamma_{gi}}(G)\) – set.

**Example 3.2.** For the graph G given Figure 5, S = \{v_1, v_2, v_3, v_5\} is the unique minimum geodetic certified dominating set of G and so \(c_{\gamma_{g}}(G) = 4\). Here the induced subgraph \(\langle S \rangle\) is not independent.
Now it is clear that $S_1 = \{v_1, v_5, v_9, v_{10}, v_{11}\}$ and $S_2 = \{v_2, v_3, v_6, v_7, v_8\}$ are the minimum independent geodetic certified dominating set of $G$ and so $c\gamma_{gi}(G) = 5$. Thus the geodetic certified domination number and the independent geodetic certified domination number are different.

**Observation 3.3.** Let $G$ be a connected graph. Then

(i) Every extreme vertex belong to every independent geodetic certified dominating set of $G$.

(ii) If $G$ contains at least two adjacent extreme vertices, then $G$ has no independent geodetic certified dominating set.

(iii) All graphs do not possess independent geodetic certified dominating set. Example $K_p$ and $C_5$ have no independent geodetic certified dominating set.

(iv) If $G$ contains a pendent vertex, then $G$ has no independent geodetic certified dominating set.

Let $\zeta$ denote the collection of all graphs having at least one independent geodetic certified dominating set.

**Observation 3.4.** Let $G \in \zeta$. Then the following are observed.

(i) $2 \leq c\gamma_g(G) \leq c\gamma_{gi}(G) \leq \frac{p}{2}$.

(ii) If $S$ is a minimum independent geodetic certified dominating set of $G$, then $V(G) - S$ is a dominating set of $G$.

**Theorem 3.5.** Let $G \in \zeta$ be a connected graph of order $p \geq 3$. Then $c\gamma_{gi}(G) = 2$ if and only if $c\gamma_g(G) = 2$.

**Proof.** Let $G \in \zeta$. Assume $c\gamma_{gi}(G) = 2$. Then by observation 3.4 (i), $c\gamma_g(G) = 2$. Conversely, assume $c\gamma_g(G) = 2$. Let $S = \{x, y\}$ be a minimum geodetic certified dominating set of $G$. Since $p \geq 3$, that $G \neq K_2$ and so $d(x, y) \geq 2$. Then $\langle S \rangle$ is independent geodetic certified dominating set of $G$ and so $c\gamma_{gi}(G) \leq |S| = 2$. Hence, $c\gamma_{gi}(G) = 2$.

**Theorem 3.6.** For $n \geq 6$, $c\gamma_{gi}(C_p) = c\gamma_g(C_p) = \left\lceil \frac{p}{3} \right\rceil$.

**Proof.** Let $C_p = (v_1, v_2, ..., v_p)$. If $p \equiv 0 \pmod{3}$ or $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$, then $S = \{v_1, v_4, ..., v_{p-2}\}$ or $S = \{v_1, v_2, v_6, v_9, ..., v_{p-4}, v_{p-2}\}$ or $S = \{v_1, v_4, ..., v_{p-4}, v_{p-1}\}$ is a minimum geodetic certified dominating set of $C_p$ and so $c\gamma_g(C_p) = \left\lceil \frac{p}{3} \right\rceil$. Since $\langle S \rangle$ is an independent set, we conclude $c\gamma_{gi}(C_p) = c\gamma_g(C_p) = \left\lceil \frac{p}{3} \right\rceil$.

**Theorem 3.7.** For $p, q \geq 2$, $c\gamma_{gi}(K_{p,q}) = \min\{p, q\}$.

**Proof.** Let $G = K_{p,q}$. Let $X = \{x_1, x_2, ..., x_p\}$ and $Y = \{y_1, y_2, ..., y_q\}$ be a bipartition of $G$ with $|X| = p$ and $|Y| = q$. Since every vertices in $X$ is adjacent to a vertex in $Y$, no independent geodetic certified dominating set contain vertices from both $X$ and $Y$. Let $S$ be a minimum independent geodetic certified dominating set of $G$. Since $\langle S \rangle$ is independent, $S = X$ or $Y$. Then $c\gamma_{gi}(K_{p,q}) = \min\{p, q\}$.

**Remark 3.8.**
(i) Independent geodetic certified dominating does not exists if G is tree.
(ii) Independent geodetic certified dominating does not exists if every vertex of G is either an extreme vertex or support vertex.

**Open Problem:** Characterize the graph for which independent geodetic certified dominating set exists.

**References**


