SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY USING AL-OBBOUDI DIFFERENTIAL OPERATOR

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Abstract: In this paper a new subclass \( \Psi_\alpha^n(\alpha, \beta, \gamma, \delta, \sigma, c) \) of functions \( f(z) = z - \sum_{j=2}^{\infty} a_j z^j, a_j \geq 0, j = 2,3,... \) which are normalized and univalent in unit disk \( U = \{ z : |z| < 1 \} \) is defined. Among other results we have determined certain characterization of \( \Psi_\alpha^n(\alpha, \beta, \gamma, \delta, \sigma, c) \), distortion theorem, radius of convexity. We have proved that the class \( \Psi_\alpha^n(\alpha, \beta, \gamma, \delta, \sigma, c) \) is convex. The results obtained are shown to be sharp.

Index Terms - Starlike function, distortion theorem, radius of convexity.

MSC: 30C45; 30A20

I. Introduction

Let \( S \) denote the class of analytic functions, that are univalent in unit disk \( U = \{ z : |z| < 1 \} \) and satisfy conditions of normalization \( f(0) = 0, f'(0) = 1 \). Let \( T \) denote the subclass of \( S \) consisting of functions whose non-zero coefficients, from the second on, are negative. Thus, analytic and univalent function \( f \) is in \( T \) if and only if it can be expressed in the form

\[
  f(z) = z - \sum_{j=2}^{\infty} a_j z^j \quad a_j \geq 0, j = 2,3,...
\]

We denote \( N_0 \), the set of all non-negative integers (\( N_0 = \{ 0,1,2,... \} \)). Differential and integral operators of normalized and analytic functions are very useful. Ruscheweyh [11] in 1975 and Salagean [13] in 1983 defined and studied differential and integral operators. Various authors have been generalized and studied these two operators. In 2004 Al-Oboudi [1] generalized Salagean operator. In this paper using Al-Oboudi operator and \( \sigma^{th} \) order polylogarithm function, new subclass of \( S \) is defined and obtained coefficient inequality, distortion theorem and radius of convexity.

The Hadamard product (or convolution) of two power series \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) and \( g(z) = z + \sum_{j=2}^{\infty} b_j z^j \) is given by

\[
  (f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j
\]

Let \( \phi_{\sigma}(c, z) \) denote the generalization of the Riemann zeta and polylogarithm functions, or simply the \( \sigma^{th} \) order polylogarithm function, given by

\[
  \phi_{\sigma}(c, z) = \sum_{j=1}^{\infty} \frac{z^j}{(j + c)^{\sigma}}
\]

where any term with \( j + c = 0 \) is excluded (see Lerch [2]). Using the definition of the Gamma function a simple transformation produces the integral formula

\[
  \phi_{\sigma}(c, z) = \frac{1}{\Gamma(\sigma)} \int_0^1 \left( \frac{\log \left( \frac{t}{1-t} \right)^{\sigma-1}}{t^{c+1}} - 1 \right) dt,
\]

where \( \text{Re}(c) > -1 \) and \( \text{Re}(\sigma) > 1 \).
Definition 1 [3]. For $f \in S$, Al-Oboudi [1], defined an operator $D^D_f: S \rightarrow S$, $\delta \geq 0, n \in N_0$, by

$$D^n_\delta f(z) = f(z).$$

$$D^1_\delta f(z) = (1 - \delta) f(z) + \delta f'(z).$$

$$D^n_\delta f(z) = D^n_\delta(D^{n-1}_\delta f(z)).$$

For $f(z)$ given by (1.1), we have

$$D^n_\delta f(z) = z - \sum_{j=2}^{\infty} \left[1 + (j-1)\delta \right]^n a_j z^j. \quad (\delta \geq 0, n \in N_0)$$

(1.2)

For $\delta = 1$, we obtain Salagean derivative operator [13].

Authors like Obradovic and Joshi [10], Salagean [12], Naik and Chougule [2], Joshi et. al. [5,6] and Sangle et al. [7], have investigated various subclasses of analytic functions. Motivated by aforementioned work, we introduce a new subclass $\Psi_n^e(\alpha, \beta, \gamma, \delta, \sigma, c)$.

Definition 2. Let $\alpha \in \{0, 1\}$, $\beta \in \{0, 1\}$, $\gamma \in \{1/2, 1\}$, and let $c$, we define, the class denoted

$$\Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c) = \{ f \in S : f(0) = f'(0) - 1 = 0 \text{ and } |l_{n,\delta,\sigma,\alpha}(f,\psi,\alpha,\gamma; z)| < \beta, \ z \in U \}$$

where

$$l_{n,\delta,\sigma,\alpha}(f,\psi,\alpha,\gamma; z) = \frac{(D^n_\delta(f*\psi_\sigma^\epsilon)(z))'-1}{2\gamma \left( (D^n_\delta(f*\psi_\sigma^\epsilon)(z))'-1 \right) - \left( (D^n_\delta(f*\psi_\sigma^\epsilon)(z))'-1 \right)}.$$  

and $\psi_\sigma^\epsilon(z) = (1 + c)^\sigma \phi_\sigma(c, z)$.

Then, we define $\Psi_n^e(\alpha, \beta, \gamma, \delta, \sigma, c) = \Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c) \cap T$.

Remarks: i) For $n = 0, \delta = 1$, $\psi_\sigma^\epsilon(z) = z/(1 - z)$, we obtain the class $S(\alpha, \beta, \gamma)$ introduced and studied by Kulkarni [8].

ii) For $n = 0, \delta = 1, \gamma = 1$, $\psi_\sigma^\epsilon(z) = z/(1 - z)$, we obtain the class $P^e(\alpha, \beta)$ introduced and studied by Gupta and Jain [4].

iii) For $\delta = 1$, $\psi_\sigma^\epsilon(z) = z/(1 - z)$, we obtain the class $D_n(\alpha, \beta, \gamma)$ introduced and studied by Shelake [14].

II. CHARACTERIZATION OF CLASS $\Psi_n^e(\alpha, \beta, \gamma, \delta, \sigma, c)$

Theorem 1. Let $\alpha \in \{0, 1\}$, $\beta \in \{0, 1\}$, $\gamma \in \{1/2, 1\}$ and let $n \in N_0$, $\delta \geq 0, \sigma > 1, c > 0$. The function $f$ of the form (1.1) is in $\Psi_n^e(\alpha, \beta, \gamma, \delta, \sigma, c)$ if and only if

$$\sum_{j=2}^{\infty} j \left[1 + (j-1)\delta \right]^n \left[1 + \beta(2\gamma - 1) \left( \frac{1 + c}{j + c} \right) \right] a_j \leq 2\beta\gamma(1 - \alpha)$$

(2.1)

The result (2.1) is sharp.

Proof. We suppose that (2.1) holds. Then we have

$$|l_{n,\delta,\sigma,\alpha}(f,\psi,\alpha,\gamma; z)| = \left| \frac{(D^n_\delta(f*\psi_\sigma^\epsilon)(z))'-1}{2\gamma \left( (D^n_\delta(f*\psi_\sigma^\epsilon)(z))'-1 \right) - \left( (D^n_\delta(f*\psi_\sigma^\epsilon)(z))'-1 \right)} \right|.$$
\[
\sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j z^{j-1} = \frac{2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j (2\gamma - 1) z^{j-1}}{2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j (2\gamma - 1) z^{j-1}}
\]

Let \( |z| = 1 \), then
\[
\left| \sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j z^{j-1} \right| \leq \beta \cdot \left| 2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j (2\gamma - 1) z^{j-1} \right| 
\]
\[
\leq \sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left[ 1 + \beta(2\gamma - 1) \right] \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j - 2\beta \gamma(1-\alpha) \leq 0.
\]

where we used (2.1).

From the last inequality we deduce
\[
\left| l_{n,\delta,\sigma}(f,\psi,\alpha,\gamma; \alpha) \right| \leq \beta \quad \text{for} \quad |z| = 1.
\]

Hence
\[
\left| l_{n,\delta,\sigma}(f,\psi,\alpha,\gamma; \alpha) \right| < \beta \quad \text{for} \quad z \in U \quad \text{and} \quad f \in \Psi_n(\alpha,\beta,\gamma,\delta,\sigma,\alpha).
\]

Conversely we assume that \( f \in \Psi_n(\alpha,\beta,\gamma,\delta,\sigma,\alpha) \). Then
\[
\left| l_{n,\delta,\sigma}(f,\psi,\alpha,\gamma; \alpha) \right| < \beta \quad \text{for} \quad z \in U.
\]

We note that
\[
E(z) = 2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j (2\gamma - 1) z^{j-1} > 0 \quad z \in [0,1],
\]
for \( \alpha \in [0,1] \) and \( E(0) = 2\gamma(1-\alpha) > 0 \). Upon clearing the denominator in (2.3) and letting \( z \to 1 \) through real values, we deduce
\[
\sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j \leq 2\beta \gamma(1-\alpha) - \beta \sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j (2\gamma - 1) \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j.
\]

Thus
\[
\sum_{j=2}^{\infty} j \left[ 1 + (j-1) \delta \right]^m \left[ 1 + \beta(2\gamma - 1) \right] \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j \leq 2\beta \gamma(1-\alpha).
\]

The extremal functions are
\[
f_j(z) = z - \frac{2\beta \gamma(1-\alpha)}{j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j \left[ 1 + \beta(2\gamma - 1) \right]} z^j \quad j = 2,3,\ldots
\]

Corollary 2. If \( f \in \Psi_n(\alpha,\beta,\gamma,\delta,\sigma,\alpha) \) then
\[
a_j \leq \frac{2\beta \gamma(1-\alpha)}{j \left[ 1 + (j-1) \delta \right]^m \left( \frac{1+c}{j+c} \right)^\sigma \alpha_j^j \left[ 1 + \beta(2\gamma - 1) \right]} \quad j = 2,3,\ldots
\]
The result is sharp and the extremal functions are given by (2.4).

Next we obtain a theorem which supplies the extreme point of the class \( \Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c) \).

**Theorem 3.** Let \( f_j(z) = z \)
and
\[
f_j(z) = z - \frac{2 \gamma \beta (1 - \alpha)}{j[1+(j-1)\delta]^{\sigma} \left(\frac{1+c}{j+c}\right)^{\sigma} [1 + \beta (2\gamma -1)]} z^j.
\]

Then \( f \in \Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c) \) if it can be expressed in the form
\[
f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z)
\]
where \( \lambda_j \geq 0 \) \((j=1,2,3,\ldots)\) and \( \lambda_1 + \sum_{j=2}^{\infty} \lambda_j = 1 \).

**Proof.** Suppose that
\[
f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z)
\]

\[
= z - \sum_{j=2}^{\infty} \frac{2 \gamma \beta (1 - \alpha) \lambda_j}{j[1+(j-1)\delta]^{\sigma} \left(\frac{1+c}{j+c}\right)^{\sigma} [1 + \beta (2\gamma -1)]} z^j.
\]

Since
\[
\sum_{j=2}^{\infty} \frac{2 \gamma \beta (1 - \alpha) \lambda_j}{j[1+(j-1)\delta]^{\sigma} \left(\frac{1+c}{j+c}\right)^{\sigma} [1 + \beta (2\gamma -1)]} \lambda_j = 2 \beta \gamma (1 - \alpha) \sum_{j=2}^{\infty} \lambda_j \leq 2 \beta \gamma (1 - \alpha).\]

By Theorem 1 \( f \in \Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c) \).

Conversely, we suppose that \( f \in \Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c) \).
Since
\[
a_j \leq \frac{2 \gamma \beta (1 - \alpha)}{j[1+(j-1)\delta]^{\sigma} \left(\frac{1+c}{j+c}\right)^{\sigma} [1 + \beta (2\gamma -1)]} j=2,3,\ldots.
\]

Setting
\[
\lambda_j = \frac{j[1+(j-1)\delta]^{\sigma} \left(\frac{1+c}{j+c}\right)^{\sigma} [1 + \beta (2\gamma -1)]}{2 \gamma \beta (1 - \alpha) a_j}
\]
and
\[
\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j.
\]

Then we have
\[
f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z).
\]
This completes the proof of Theorem 2.
Corollary 4. The extreme points of \( \Psi_n^*(\alpha, \beta, \gamma, \delta, \sigma, c) \) are the functions

\[
f_i(z) = z
\]
and

\[
f_j(z) = z - \frac{2 \gamma \beta (1 - \alpha)}{j [1 + (j - 1) \delta]^{\alpha} \left( \frac{1 + c}{j + c} \right)^\sigma [1 + \beta (2 \gamma - 1)]} z^j \quad j = 2, 3, \ldots.
\]

III. DISTORTION THEOREM

Theorem 1. Let \( \alpha \in [0, 1), \beta \in (0, 1], \gamma \in (1/2, 1] \) and let \( n \in \mathbb{N}_0, \delta \geq 0, \sigma > 1, c > 0 \), if \( f \in \Psi_n^*(\alpha, \beta, \gamma, \delta, \sigma, c) \), then for \( 0 < |z| = r < 1 \), we have

\[
r - \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^n \left[ 1 + \beta (2 \gamma - 1) \right]} r^2 \leq \left| f * \psi_{\sigma}^*(z) \right| \leq r + \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^{n-1} \left[ 1 + \beta (2 \gamma - 1) \right]} r^2
\]
and

\[
1 - \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^{-n-1} \left[ 1 + \beta (2 \gamma - 1) \right]} r \leq \left| D_s f * \psi_{\sigma}^*(z) \right| \leq 1 + \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^{-n} \left[ 1 + \beta (2 \gamma - 1) \right]} r
\]

The bounds in (3.1) and (3.2) are sharp.

Proof. From (2.1) we have

\[
2(1 + \delta)^{n+k} \left[ 1 + \beta (2 \gamma - 1) \right] \sum_{j=2}^{\infty} j (1 + (j - 1) \delta)^{k} \left( \frac{1 + c}{j + c} \right)^\sigma a_j \leq \sum_{j=2}^{\infty} j (1 + (j - 1) \delta)^{n} \left( \frac{1 + c}{j + c} \right)^\sigma \left[ 1 + \beta (2 \gamma - 1) \right] a_j \leq 2 \beta \gamma (1 - \alpha) \sum_{j=2}^{\infty} (1 + (j - 1) \delta)^{k} \left( \frac{1 + c}{j + c} \right)^\sigma a_j \leq \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^{n+k} \left[ 1 + \beta (2 \gamma - 1) \right]} .
\]

Using (3.3) with \( k = 0 \), for \( 0 < |z| = r < 1 \) we obtain,

\[
\left| f * \psi_{\sigma}^*(z) \right| \leq r + \sum_{j=2}^{\infty} \left( \frac{1 + c}{j + c} \right)^\sigma a_j r^j \leq r + r^2 \sum_{j=2}^{\infty} \left( \frac{1 + c}{j + c} \right)^\sigma a_j \leq r + \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^n \left[ 1 + \beta (2 \gamma - 1) \right]} r^2 .
\]

And

\[
\left| f * \psi_{\sigma}^*(z) \right| \geq r - \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^n \left[ 1 + \beta (2 \gamma - 1) \right]} r^2 .
\]

Similarly using (3.3) with \( k = 1 \), for \( 0 < |z| = r < 1 \) we obtain,

\[
\left| D_s f * \psi_{\sigma}^*(z) \right| \leq 1 + r \sum_{j=2}^{\infty} (1 + (j - 1) \delta)^{k} \left( \frac{1 + c}{j + c} \right)^\sigma a_j \leq 1 + \frac{\beta \gamma (1 - \alpha)}{(1 + \delta)^{n-1} \left[ 1 + \beta (2 \gamma - 1) \right]} r .
\]
\[ D_\delta\left( f * \psi_\sigma \right)(z) \geq 1 - \frac{\beta \gamma (1-\alpha)}{(1+\delta)^{n-1} [1 + \beta (2\gamma -1)]} r. \]

This completes the proof of Theorem 3. Sharpness are attained by the function

\[ f(z) = z - \frac{\beta \gamma (1-\alpha)}{(1+\delta)^{n}} [1 + \beta (2\gamma -1)] z^2 \quad (z = \pm r) \] (3.4)

**Theorem 2.** The disk \(|z| < 1\) is mapped onto a domain that contains the disk

\[ |w| < 1 - \frac{\gamma \beta (1-\alpha)}{(1+\delta)^{n}} [1 + \beta (2\gamma -1)] \]

by any \( f * \psi_\sigma^c \) where, \( f \in \Psi'(\alpha, \beta, \gamma, \delta, \sigma, c) \).

**Proof.** The result follows upon by letting \( r \to 1 \) in (3.1).

**IV. RADIUS OF CONVEXITY**

**Theorem 1.** If the function \( f \in \Psi'(\alpha, \beta, \gamma, \delta, \sigma, c) \), then \( f \) is convex in the disk

\[ |z| < r = r (\alpha, \beta, \gamma, \delta, \sigma, c n) = \inf_j \left( \left[ \frac{1+(j-1)\delta^n}{2j\beta\gamma(1-\alpha)} \left( \frac{1+c}{j+c} \right)^\sigma \right]^{1/j} \right) \quad (j = 2, 3, \ldots). \] (4.1)

This result is sharp, with the extremal function as given in (2.4).

**Proof.** It suffices to show that

\[ \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 \quad \text{in} \quad |z| \leq r (\alpha, \beta, \gamma, \delta, \sigma, c n) \] (4.2)

In view of (1.1) we have

\[ \left| \frac{z f''(z)}{f'(z)} \right| \leq \frac{\sum_{j=2}^\infty j (j-1) a_j |z|^{j-1}}{1 - \sum_{j=2}^\infty j a_j |z|^{j-1}}. \]

Thus (4.2) follows if

\[ \sum_{j=2}^\infty j (j-1) a_j |z|^{j-1} \leq 1 - \sum_{j=2}^\infty j a_j |z|^{j-1} \]

or

\[ \sum_{j=2}^\infty j^2 a_j |z|^{j-1} \leq 1. \] (4.3)

Also by Theorem 1, we have

\[ \sum_{j=2}^\infty j \left[ \frac{1+(j-1)\delta^n}{2\beta\gamma(1-\alpha)} \left( \frac{1+c}{j+c} \right)^\sigma \right] a_j \leq 1. \] (4.4)

Hence \( f \) is convex if

\[ j^2 |z|^{j-1} \leq \frac{j \left[ \frac{1+(j-1)\delta^n}{2\beta\gamma(1-\alpha)} \left( \frac{1+c}{j+c} \right)^\sigma \right] a_j}{1}. \]

Solving for \(|z|\), we obtain
\[ |z| \leq \left( \frac{[1+(j-1)\delta^n][1+\beta(2\gamma-1)](1+c)}{2j\beta\gamma(1-\alpha)} \right)^{1/(j-1)} (j=2,3,...). \] (4.5)

Setting \(|z|=r\) \((\alpha, \beta, \gamma, \delta, \sigma, c, n)\) in (4.2), the result follows.

**Theorem 2.** The class \(\Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c)\) is convex.

**Proof.** Let \(f_1(z) = z - \sum_{j=2}^{\infty} a_j z^j\) and \(f_2(z) = z - \sum_{j=2}^{\infty} b_j z^j\) be in \(\Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c)\).

For \(0 \leq \lambda \leq 1\), we shall prove that \(F(z) = \lambda f_1(z) + (1-\lambda) f_2(z)\) is also in class \(\Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c)\).

Since for \(0 \leq \lambda \leq 1\),

\[ F(z) = z - \sum_{j=2}^{\infty} \left[ \lambda a_j + (1-\lambda) b_j \right] z^j. \] (4.6)

We observe that

\[
\sum_{j=2}^{\infty} j \left[1+(j-1)\delta^n\right] \left[1+\beta(2\gamma-1)\right] \left(\frac{1+c}{j+c}\right)^\sigma \left\{ \lambda a_j + (1-\lambda) b_j \right\} \\
= \lambda \sum_{j=2}^{\infty} j \left[1+(j-1)\delta^n\right] \left[1+\beta(2\gamma-1)\right] \left(\frac{1+c}{j+c}\right)^\sigma a_j \\
+ (1-\lambda) \sum_{j=2}^{\infty} j \left[1+(j-1)\delta^n\right] \left[1+\beta(2\gamma-1)\right] \left(\frac{1+c}{j+c}\right)^\sigma b_j \\
\leq 2\beta\gamma(1-\alpha).
\]

Hence \(F(z) \in \Psi_n(\alpha, \beta, \gamma, \delta, \sigma, c)\). This completes the proof of Theorem 6.

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