ON SUBCLASSES OF UNIVALENT FUNCTION DEFINED USING GENERALIZED DIFFERENTIAL AND INTEGRAL OPERATOR.

1 V. A. Chougule, 2 U. H. Naik

1Assist. Prof. Department of Mathematics, College of Engineering Malegaon Bk.Baramati, Pune, India.

,3Head, Department of Mathematics, Willingdon College, Sangli India

Abstract : The subclasses of univalent functions are defined using generalized differential operator studied by M. Darus and R.W. Ibrahim [1] which is generalization of well known Salagean Operator. The properties of these classes are studied in this paper.

IndexTerms - univalent functions, Salagean, Differential operator, radii, Starlike, Convex.

I. INTRODUCTION, DEFINITION AND MOTIVATION

Let A denotes the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]  \hspace{1cm} (1.1)

which are analytic in the unit disk \( U = \{ z : |z| < 1 \} \).

Let \( S \) be subclass of \( A \), of functions univalent in \( U \).

For the function (1.1), we define following generalized differential operator.

\[ \mathcal{D}^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

For \( a, b \geq 0 \) and \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \),

\[ \mathcal{D}^k_{a,b} f(z) = \mathcal{D}^1_{a,b} [\mathcal{D}^{k-1}_{a,b} f(z)] = z + \sum_{n=2}^{\infty} [n + (n - 1)(b - a)]^k a_n z^n = z + \sum_{n=2}^{\infty} A^k a_n z^n \]

. We write \( A \) for \( [n + (n - 1)(b - a)] \) throughout this article.

This generalized differential operator is defined and studied by Darus and Ibrahim [1] and for \( a = 1 \) and \( b = \lambda \) we get generalized Salagean Operator [2, 3] and for \( a = b \), the Salagean Operator.(See [4, 5, 6, 7, 10, 11, 12] and many others).

Definition 1.1 : The function \( f(z) \) defined by Equation (1.1), is said to be in the class \( \mathcal{S}^k_{a,b}(\lambda, \alpha, \beta) \) for \( |\alpha| \leq 1 \) and \( \lambda, \beta \geq 0 \),

\[ \Re \left\{ \frac{z [\mathcal{D}^k_{a,b} f(z)]} {\mathcal{D}^k_{a,b} f(z)} \left[ \frac{z [\mathcal{D}^k_{a,b} f(z)]} {\mathcal{D}^k_{a,b} f(z)} \right]' \right\} \geq \beta \left| z [\mathcal{D}^k_{a,b} f(z)] \right|^\lambda + \lambda \alpha \left| \frac{z [\mathcal{D}^k_{a,b} f(z)]} {\mathcal{D}^k_{a,b} f(z)} \right|^\alpha - 1 \} \hspace{1cm} (1.2) \]

Definition 1.2 : The function \( f(z) \) defined by Equation (1.1), is said to be in the class \( \mathcal{C}^k_{a,b}(\lambda, \alpha, \beta) \) for \( |\alpha| \leq 1 \) and \( \lambda, \beta \geq 0 \),

\[ \Re \left\{ 1 + \frac{z [\mathcal{D}^k_{a,b} f(z)]} {\mathcal{D}^k_{a,b} f(z)} \left[ \frac{z [\mathcal{D}^k_{a,b} f(z)]} {\mathcal{D}^k_{a,b} f(z)} \right]' \right\} \geq \beta \left| z [\mathcal{D}^k_{a,b} f(z)] \right|^\lambda + \lambda \alpha \left| \frac{z [\mathcal{D}^k_{a,b} f(z)]} {\mathcal{D}^k_{a,b} f(z)} \right|^\alpha - 1 \} \hspace{1cm} (1.3) \]

Remark: Above subclasses are generalization of many others (see [8]).

- For \( k = 0 \) we get results of Chougule and Naik (see [9]).
  (a) \( \mathcal{S}^0_{a,b}(\lambda, \alpha, \beta) \equiv \text{class } S(\lambda, \alpha, \beta) \).
  (b) \( C^0_{a,b}(\lambda, \alpha, \beta) \equiv \text{class } C(\lambda, \alpha, \beta) \).
- For \( k = 0 \) and \( \lambda = 0 \)
  (a) \( \mathcal{S}^0_{a,b}(0, \alpha, \beta) \equiv \text{class } \beta - \text{uniformly starlike functions of order } \alpha \).
  (b) \( \mathcal{C}^0_{a,b}(0, \alpha, \beta) \equiv \text{class } \beta - \text{uniformly convex functions of order } \alpha \).
II. INCLUSION PROPERTY

In this section we will obtain the conditions for the Function \( f(z) \) to be in the subclasses \( S_{a,b}^k(\lambda, \alpha, \beta) \) and \( C_{a,b}^k(\lambda, \alpha, \beta) \), by using coefficient inequalities and generalized differential operator.

**Theorem 2.1** The function \( f(z) \) defined by (1.1) is in the subclass \( S_{a,b}^k(\lambda, \alpha, \beta) \) if and only if,

\[
\sum_{n=1}^{\infty} [(\beta - 1)(n + \lambda n(n-1)) + \alpha - \beta] A^k |a_n| \leq 1 - \alpha. \tag{2.1}
\]

**PROOF:** Using Definition of generalized differential operator for the function defined by (1.1), we can write,

\[
\frac{z[D_{a,b}^k f(z)]'' + \lambda z^2[D_{a,b}^k f(z)]'''}{D_{a,b}^k f(z)} - 1 = \frac{\sum_{n=2}^{\infty} (n + \lambda n^2 - \lambda n - 1) A^k a_n z^n}{z + \sum_{n=2}^{\infty} A^k a_n z^n}.
\]

Hence the Inequality (1.2) of definition 1.1 is written as,

\[
\beta \left[ \sum_{n=2}^{\infty} \frac{(n + \lambda n^2 - \lambda n - 1) A^k a_n z^n}{z + \sum_{n=2}^{\infty} A^k a_n z^n} \right] \leq Re\left\{ \sum_{n=2}^{\infty} \frac{(n + \lambda n^2 - \lambda n - 1) A^k a_n z^n}{z + \sum_{n=2}^{\infty} A^k a_n z^n} \right\} + 1 - \alpha.
\]

Considering the fact that, \( Re(z) \leq |z| \)

\[
Re\{(\beta - 1) \sum_{n=2}^{\infty} (n + \lambda n^2 - \lambda n - 1) A^k a_n z^n\} \leq (1 - \alpha) Re\{z + \sum_{n=2}^{\infty} A^k a_n z^n\}.
\]

For the function \( f(z) \) defined in unit disk \( U = \{z : |z| < 1\} \), letting \( z \to 1_- \), we obtain (2.1).

**Theorem 2.2** The function \( f(z) \) defined by (1.1) is in the subclass \( C_{a,b}^k(\lambda, \alpha, \beta) \) if and only if,

\[
\sum_{n=1}^{\infty} n[(\beta - 1)(n + \lambda(n-1)(n-2)) + \alpha - \beta] A^k |a_n| \leq 1 - \alpha. \tag{2.2}
\]

**PROOF:** We have,

\[
\frac{z[D_{a,b}^k f(z)]'''}{D_{a,b}^k f(z)} = \frac{\sum_{n=2}^{\infty} n(n - 1) [1 + \lambda(n-2)] A^k a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n A^k a_n z^{n-1}}.
\]

Therefore the Inequality (1.3) of definition 1.2,

\[
\beta \left[ \sum_{n=2}^{\infty} \frac{n(n - 1) [1 + \lambda(n-2)] A^k a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n A^k a_n z^{n-1}} \right] \leq Re\left\{ \sum_{n=2}^{\infty} \frac{n(n - 1) [1 + \lambda(n-2)] A^k a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n A^k a_n z^{n-1}} \right\} + 1 - \alpha.
\]

As \( Re(z) \leq |z| \) and letting \( z \to 1_- \), we obtain (2.2).

For \( \lambda = 0 \), we can easily write the following results.

**Corollary 2.1** The function, \( F(z) = D_{a,b}^k f(z) \) is \( \beta \) - uniformly starlike functions of order \( \alpha \) if and only if

\[
\sum_{n=1}^{\infty} n[(\beta - 1) + \alpha - \beta] A^k |a_n| \leq 1 - \alpha
\]

**Corollary 2.2** The function, \( F(z) = D_{a,b}^k f(z) \) is \( \beta \) - uniformly convex functions of order \( \alpha \) if and only if

\[
\sum_{n=1}^{\infty} n[(\beta - 1) + \alpha - \beta] A^k |a_n| \leq 1 - \alpha
\]

III. DISTORTION AND GROWTH BOUNDS

3.1. DISTORTION AND GROWTH BOUNDS FOR \( S_{a,b}^k(\lambda, \alpha, \beta) \)

The following theorems gives Distortion and Growth Bounds for \( S_{a,b}^k(\lambda, \alpha, \beta) \).

**Theorem 3.1** If \( f(z) \in S_{a,b}^k(\lambda, \alpha, \beta) \) as defined in (1.1) then for \( |z| = r < 1 \),

\[
r \left[ \frac{1 - \alpha}{(2 - a + b)^k [2\lambda(\beta - 1) + \alpha + \beta - 2]} \right]^r \leq |f(z)| \leq r + \frac{1 - \alpha}{(2 - a + b)^k [2\lambda(\beta - 1) + \alpha + \beta - 2]} \left[ \frac{1 - \alpha}{(2 - a + b)^k [2\lambda(\beta - 1) + \alpha + \beta - 2]} \right]^r \tag{3.1}
\]

\[
r \left[ \frac{2(1 - \alpha)}{(2 - a + b)^k [2\lambda(\beta - 1) + \alpha + \beta - 2]} \right]^r \leq |f''(z)| \leq 1 + \frac{2(1 - \alpha)}{(2 - a + b)^k [2\lambda(\beta - 1) + \alpha + \beta - 2]} \left[ \frac{2(1 - \alpha)}{(2 - a + b)^k [2\lambda(\beta - 1) + \alpha + \beta - 2]} \right]^r \tag{3.2}
\]
3.2. Distortion And Growth Bounds For $C_{a,b}^k(\lambda, \alpha, \beta)$

Theorem 3.2 If $f(z) \in C_{a,b}^k(\lambda, \alpha, \beta)$ as defined in (1.1) then for $|z| = r < 1$,

$$r - \frac{1}{2(a + b)(\alpha + \beta - 2)} z^2 \leq |f(z)| \leq r + \frac{1}{2(a + b)(\alpha + \beta - 2)} z^2$$  \hspace{1cm} (3.3)

$$1 - \frac{1}{2(a + b)(\alpha + \beta - 2)} 2r \leq |f'(z)| \leq 1 + \frac{1}{2(a + b)(\alpha + \beta - 2)} 2r$$  \hspace{1cm} (3.4)

IV. RADIUS OF STARLIKENESS AND CONVEXITY

4.1. For The Subclass $S_{a,b}^k(\lambda, \alpha, \beta)$

Theorem 4.1 If $f(z) \in S_{a,b}^k(\lambda, \alpha, \beta)$ as defined in (1.1) then $f(z)$ is starlike of order $\delta(0 \leq \delta < 1)$ in the disk $|z| < r_2$ where,

$$r_2 = \inf_{n \geq 2} \left[ \frac{(1-\delta)(\beta-1)(n+\lambda n(n-1)+\alpha-\beta)A^k}{n(n-2+\delta)(1-\alpha)} \right]^{n-1}$$ \hspace{1cm} (4.1)

Theorem 4.2 Let the function $f(z)$ defined by Equation (1.1) belongs to $S_{a,b}^k(\lambda, \alpha, \beta)$, then $f(z)$ is convex of order $\delta(0 \leq \delta < 1)$ in the disk $|z| < r_3$ where,

$$r_3 = \inf_{n \geq 2} \left[ \frac{(1-\delta)(\beta-1)(n+\lambda n(n-1)+\alpha-\beta)A^k}{n(n-2+\delta)(1-\alpha)} \right]^{n-1}$$ \hspace{1cm} (4.2)

4.2. For The Subclass $C_{a,b}^k(\lambda, \alpha, \beta)$

Theorem 4.3 If $f(z) \in C_{a,b}^k(\lambda, \alpha, \beta)$ as defined in (1.1) then $f(z)$ is starlike of order $\delta(0 \leq \delta < 1)$ in the disk $|z| < r_2$ where,

$$r_2 = \inf_{n \geq 2} \left[ \frac{(1-\delta)(\beta-1)(n+\lambda n(n-1)+\alpha-\beta)A^k}{n(n-2+\delta)(1-\alpha)} \right]^{n-1}$$ \hspace{1cm} (4.3)

Theorem 4.4 Let the function $f(z)$ defined by Equation (1.1) belongs to $C_{a,b}^k(\lambda, \alpha, \beta)$, then $f(z)$ is convex of order $\delta(0 \leq \delta < 1)$ in the disk $|z| < r_3$ where,

$$r_3 = \inf_{n \geq 2} \left[ \frac{(1-\delta)(\beta-1)(n+\lambda n(n-1)+\alpha-\beta)A^k}{n(n-2+\delta)(1-\alpha)} \right]^{n-1}$$ \hspace{1cm} (4.4)

V. INTEGRAL OPERATOR

For, $F(z) = z + \sum_{n=2}^{\infty} [n + (n-1)(b-a)]^k z^n = z + \sum_{n=2}^{\infty} A^k z^n$ define,

$$[F(z)]^{-1} = z + \sum_{n=2}^{\infty} \frac{n}{[n + (n-1)(b-a)]^k} z^n = z + \sum_{n=2}^{\infty} \frac{n}{A^k} z^n$$

so that the convolution,

$$F(z) * [F(z)]^{-1} = z + \sum_{n=2}^{\infty} n z^n = \frac{z}{(1-z)^2}$$

Now define the Integral operator as follows,

For the function $f(z)$ defined by Equation (1.1),

$$I_{a,b}^k f(z) = [F(z)]^{-1} * f(z) = z + \sum_{n=2}^{\infty} \frac{n}{[n + (n-1)(b-a)]^k} a_n z^n = z + \sum_{n=2}^{\infty} \frac{n}{A^k} a_n z^n$$ \hspace{1cm} (5.1)

REMARK

1. For $a = b$, generalized integral operator (5.1) reduces to the integral operator,

$$I_{a,b}^k f(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{n^{k-1}} z^n$$

2. For $k = 0$, $I_{a,b}^0 f(z) = z + \sum_{n=2}^{\infty} n a_n z^n = z f'(z)$. 

3. For $k = 1$ and $a = b$, $I_{a,a}^1 f(z) = f(z)$.

4. For $k = 2$ and $a = b$, $I_{a,a}^2 f(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{n^2} z^n = \int_0^z f(t) \, dt$

Definition 5.1: The function $f(z)$ defined by Equation (1.1), is said to be in the class $S_{a,b}^k(\lambda, \alpha, \beta)$ for $|a| \leq 1$ and $\lambda, \beta \geq 0$,

$$Re \left\{ \frac{z [I_{a,b}^k f(z)]'}{I_{a,b}^k f(z)} - \alpha \right\} \geq \beta \left| \frac{z [I_{a,b}^k f(z)]'}{I_{a,b}^k f(z)} - 1 \right|$$ \hspace{1cm} (5.2)
Definition 5.2: The function $f(z)$ defined by Equation (1.1), is said to be in the class $\mathcal{C}_{\alpha,\beta}(\lambda, \alpha, \beta)$ for $|\alpha| \leq 1$ and $\lambda, \beta \geq 0$.

\[ \Re \left\{ 1 + z \frac{[g_{\alpha,\beta}^k f(z)]^{''''}}{[g_{\alpha,\beta}^k f(z)]''} + \lambda z^2 \frac{[g_{\alpha,\beta}^k f(z)]^{''''}}{[g_{\alpha,\beta}^k f(z)]''} \right\} \geq \beta \left\{ z \frac{[g_{\alpha,\beta}^k f(z)]^{''''}}{[g_{\alpha,\beta}^k f(z)]''} + \lambda z^2 \frac{[g_{\alpha,\beta}^k f(z)]^{''''}}{[g_{\alpha,\beta}^k f(z)]''} \right\}. \] (5.3)

The conditions for the Function $f(z)$ to be in the subclasses $\mathcal{S}_{\alpha,\beta}(\lambda, \alpha, \beta)$ and $\mathcal{C}_{\alpha,\beta}(\lambda, \alpha, \beta)$, will be obtained by using coefficient inequalities and integral operator.

Theorem 5.1: The function $f(z)$ defined by (1.1) is in the subclass $\mathcal{S}_{\alpha,\beta}(\lambda, \alpha, \beta)$ if
\[ \sum_{n=1}^{\infty} n \frac{|a_n|}{A_k} ((\beta - 1)(n + \lambda(n - 1)) + \alpha - \beta) \leq 1 - \alpha. \] (5.4)

Theorem 5.2: The function $f(z)$ defined by (1.1) is in the subclass $\mathcal{C}_{\alpha,\beta}(\lambda, \alpha, \beta)$ if
\[ \sum_{n=1}^{\infty} n^2 \frac{|a_n|}{A_k} ((\beta - 1)(n + \lambda(n - 1)(n - 2)) + \alpha - \beta) \leq 1 - \alpha. \] (5.5)

REFERENCES