A NOTE ON MOLECULAR POSETS

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Abstract: Wang Guo-Jun has introduced topological molecular lattices and he studied properties of molecular lattices. The aim of this paper is to introduce this concept to larger class say Poset and study these results in posets.

Index Terms - Complete lattice, distributive lattices, complete poset, Completely distributive poset.

I. INTRODUCTION

Wang Guo-Jun has introduced topological molecular lattices and he studied properties of molecular lattices. The aim of this paper is to introduce this concept to larger class say Poset and study these results in posets. We first introduce the concept of molecule for completely distributive complete poset, the cotopology on it. We also defined molecular posets and proved some results.

II. PRELIMINARIES

We give here some definitions and notations for ready reference; see also.

Definition 2.1. Let P be a poset. Every subset \( H \subseteq P \), \( H \neq \emptyset \) and \( \forall h_1 \in H \exists \ h_2 \in H\) such that \( h_2 \leq h_1 \), then P is called U-complete poset.

Dually we have concept of L-complete poset. If P is both U-complete and L-complete then P is called complete poset.

We have already known the role of the concept of join-density and the way it can be applied to construct some important classes of lattices as well as posets. We take the concepts defined by Alireza and Kharat for ready reference as follows.

Definition 2.2. Let P be a poset. A subset \( S \subseteq P \) is called U-dense if each element of P belongs to \( S_i \) for some \( S_i \subseteq S \). Observe that \( S \) is U-dense in \( P \) if and only if for any two elements \( a, b \in P \) with \( a < b \), there is some \( s \in S \) with \( s \leq a \) and \( s \leq b \). We also have the concept of L-density which can be defined dually.

Observe that every join-dense subset of a poset \( P \) is U-dense but the converse need not be true. An example would be atomistic posets and atomisticity in the sense of Shewale. For a given pair of elements \( a, b \) of a poset \( P \) we may have \( (a, b)^u \) is non-empty but \( (a, b)^l \) is an empty set (Fig. 1). This observation, along with U-density lead us to define completeness, compactness and compact generation in posets as follows.

Figure 1

Definition 2.3. A poset \( P \) is called U-complete if for every subset \( H \subseteq P \), \( H \neq \emptyset \) and for every \( u \in H^u \) there exists an element \( v \in H^l \) such that \( v \leq u \). A poset \( P \) is called L-complete if for every subset \( H \subseteq P \), \( H \neq \emptyset \) and for every \( l \in H^l \) there exists an element \( l \in H^u \) such that \( l \leq u \). A poset \( P \) is called complete as well as L-complete.

Observe that \( P \) is up-complete but it is not U-complete nor chain-complete. In fact, \( (a, b)^u \) is non-empty however \( (a, b)^l \) is an empty set. Also observe that it is not chain-complete as the meet of the infinite decreasing chain does not exist.

A subset \( A \) of a poset \( L \) is called relatively directed if for every pair \( a, b \) in \( A \) there exists \( x \in L \) such that \( a, b \leq x \).

A poset \( P \) is said to be distributive if, for all \( a, b, c \in P \), \( (a, b)^n \), \( c \) \( = \{ (a, c)^n, (b, c)^n \}^{ul} \) holds.

Distributivity in general: \( a_i \in P \), \( \{ (a_i)^n \} = \{ (a_i)^l \}^{ul} \)

\( \{ (a_i, j \in I)^n, i \in I \} = \{ (a_i, j \in I)^l, f \in \prod_{i \in I} \}^{ul} \)

A poset is called a completely distributive poset (CDP) if

- (CDP1) every nonempty set has an infimum
- (CDP2) every relatively directed set has suprimum and
- (CDP3) the distributive law holds for arbitrary inf and arbitrary relatively directed sups.
Hence there exists a minimal family of $a$.

Proof. Let $P = [0, 1]$ be the lattice (poset) with the order of the numbers. Then $\forall a \in \{0, 1\}, \{0, a\}$ is a minimal family of $a$ and $\{0\}$ is the minimal family of $0$.

Let $P = 2^\mathbb{N}$ be the poset with the order of set inclusion $\subseteq$. Where $X$ is a nonempty set. Then $\forall E \in P, E \subset X, \{\{e\}/e \in E\}$ is a minimal family of $E$ and $\emptyset$ is the minimal family of $\emptyset$.

Lemma 3.2. (Principle of minimal and maximal) Let $(P \leq)$ be poset. For any family $\{a_i/j \in J_i\} (i \in I, x_i \in P, I$ and $J_i$ are indexing sets) $\{\{a_i/j \in J_i\}/i \in I\} \rightarrow \{\{a_i/j \in J_i\}/f \in \prod_{i \in I} J_i\}$.

Proof. Let $x \in \{\{a_i/j \in J_i\}/i \in I\} \rightarrow \{\{a_i/j \in J_i\}/f \in \prod_{i \in I} J_i\}$ implies that $x \leq y; \forall y \in \{\{a_i/j \in J_i\}/i \in I\} \rightarrow \{\{a_i/j \in J_i\}/f \in \prod_{i \in I} J_i\}$

This is the required.

Proposition 3.3. In complete poset, (CD1) is equivalent to (CD2).

Proof of this is just showing set inclusions.

Theorem 3.4. Let $P$ be a complete poset, $a \in P$. Then the unions of minimal families of $a$ are minimal families of $a$ as well. Especially, if $a$ has a minimal family, then $a$ has a greatest minimal family.

Proof: Let $P$ be a complete poset, $a \in P, B_1, B_2, ..., B_n$ are minimal families of $a$. We claim that $\bigcup B_i$ for $i = 1, 2, ..., n$ is minimal family of $a$. $\bigcup B_i \neq \varnothing$ therefore $(\bigcup B_i) = \bigcap B_i = \bigcap_i a_i = a^*$. Since $a^* = B_i$ for $i = 1, 2, ..., n$. Hence first condition holds. Now take any subset $A \subset P$ and $a^* \subset A$. $x \in \bigcup B_i$ implies $x \in B$ for some $i$. Therefore their exist $y \in A$ such that $x \leq y$ since $B_i$ is minimal family of $a$.

As $x$ is arbitrary in $\bigcup B_i$ therefore condition second holds. Hence $\bigcup B_i$ is a minimal family of $a$.

Theorem 3.5. Let $P$ be a complete lattice. Then $P$ is CD poset if and only if $\forall a \in P, a$ has a minimal family and hence $\beta(a)$ exists.

Proof. Let $P$ be CD poset. Therefore for $a \in P, i \in I$ and $j \in J_i$ where $I, J_i$ are index sets. We have

$\{\{a_i/j \in J_i\}/i \in I\} \rightarrow \{\{a_i/j \in J_i\}/i \in I\}$

(1)

Consider $B_i = \{a_i/j \in J_i\}$ and $B = \{B_i \subset P/ i \in I, B \geq a^*\}$.

$B = \{a_i/j \in J_i\} \rightarrow \{\{a_i/j \in J_i\}/i \in I\}$

Therefore $B^{\text{up}} = \{\{a_i/j \in J_i\}/i \in I\}$

$= \{\{a_i/j \in J_i\}/i \in I\}$

$= \{a_i/j \in J_i\} \rightarrow \{a_i/j \in J_i\}$

$= a^*$, since for all $i, B_i = a_i^*$. Therefore first condition holds.

Now let $A \subset P$ and $a \subset A$. $x \in B$ with $x \leq a_i/j \forall i \in I$. For some i, $a_i = B_i$ and $a_i/j \in B_i = A$ for some i. Implies that for $x \in B_i$ there exists $a_i/j \in A$ such that $x \leq a_i/j$. Thus second condition holds.

Conversely, Let $P$ is complete poset and every element has minimal family. We claim that $P$ is complete distributive. i.e. to show $P$ satisfies either CD1 or CD2.

Let $x$ in $\{\{a_i/j \in J_i\}/i \in I\}$ then there exist $j$ in $J_i$ and $i$ in $I$ corresponding to $x$. For $j$ in $J_i$ it is easy to find $f$ in $\prod_{i \in I} J_i$ then this $x$ is in $\{\{a_i/j \in J_i\}/i \in I\}$, $f \in \prod_{i \in I} J_i$). Thus CD1 is satisfies.

Proposition 3.6. Let $P$ be a CD poset, $a, b \in P$ and $a \leq b$. Then $\beta(a) \leq \beta(b)$.

Proof. Suppose that $y \in \beta(a)$. We claim that $y \in \beta(b)$. To prove our claim we have to prove $\beta(b) \cup \{y\}$ is minimal family of $b$. Since $b \geq a \in \beth \Rightarrow \beta(b) = \beta(b) \cup \{y\}$ is minimal family of $b$. Suppose that $B \subset P$ and $b \leq B^{\text{up}}$ and $x$ is any fixed element of $\beta(b)$. If $x \in \beta(b)$, then there exists $z \in B$ such that $z \geq x$. If $x = y$, then $x \in \beta(a)$ and $a \in B^{\text{up}}$. Hence there exists $z \in B$ such that $z \geq x$. This shows that $\beta(b)$ is a minimal family of $b$ and hence $\beta(b) \subset \beta(b)$. Thus $y \in \beta(b)$. □
Proposition 3.7. Let $P$ be a CD poset and $\forall I, a_I \in P$. Then $(x' = \{a_I, I \in I\}$ and $\beta(x) = \bigcup_{I \in \beta(a)}(a_I)$.

Proof. Let $x' = \{a_I, I \in I\}$, we claim that $\beta(x) = \bigcup_{I \in \beta(a)}(a_I)$. We know that $a \leq b$ implies $\beta(a) \subseteq \beta(b)$. Therefore $\beta(a) \subseteq \beta(x)$. Since $\bigcup_{I \in \beta(a)}(a_I) = \{a_I, I \in \beta(a)\}$, $\beta(x)$ contains $a_I$ for each $I \in \beta(a)$. Hence $\beta(a) \subseteq \beta(x)$ is a lower set. Therefore $\beta(x) = \bigcup_{I \in \beta(a)}(a_I)$. \hfill \qed

Theorem 3.8. Let $P$ be a CD poset and $\beta : P \rightarrow 2^P$ be the minimal map with respect to $P$. Then following assertions hold.

1. $\beta(0) = \{0\}$
2. $\forall a \in P, \beta(a) \subseteq \beta(1)$
3. $\beta$ is a union preserving map, that is $\beta(a) = \bigcup_{I \in \beta(a)}(a_I)$.

Proof. (1) is the least element in the poset $P$. We know that $x \in \beta(a)$ implies $x \leq a$. Therefore $x \in \beta(0)$ implies that $x \leq 0$ which implies that $x = 0$ hence $\beta(0) = \{0\}$. (2) We know that if $a \leq b$ then $\beta(a) \subseteq \beta(b)$. Therefore $a \in P$ and $a < 1$ implies that $\beta(a) \subseteq \beta(1)$. Hence the second property. (3) proof of this is same as the proof of Theorem 3.7. \hfill \qed

We can define dual definition of minimal family as follows.

Definition 3.9. Let $P$ be a complete poset, $a \in P \cup P$. Then union of maximal families of $a$ are maximal families of $a$ as well. Especially, if $a$ has a maximal family, then $a$ has a greatest maximal family.

Proof of this result is the dual of the minimal family.

Theorem 3.11. Let $P$ be a complete poset. Then $P$ is CD poset if and only if $\forall a \in P$, $a$ has a maximal family and hence, $a_0(a)$ exists.

Proof of this result is the dual of the minimal family.

Lemma 3.12. Let $P$ be a CD poset, $a, b \in P$ be $a_0(a)$. Then there exists an ideal $I$ in $P$ such that

1. $a \in I (b)
2. \forall x \in P \setminus I$, there exists a minimal element $m \in P \setminus I$ such that $x \leq m$.

Proof. "Let $P$ be a CD poset, $a, b \in P$ and $b \in a_0(a)$. Then there exists a sequence $c_1, c_2, ...$ in $P$ such that $c_k \in a_0(a)$, $c_k \in a_0(c_{k-1})$, $k = 1, 2, ...$ and $b \in a_0(c_k)$. On account of conditions (1) and (2) we know that $I$ is an ideal in $P$. And $a \in I \in (b)$. Suppose $x \in P \setminus I$. Then $x \in P \setminus I$. And there exists $y \in B$ such that $y \leq x$.

We write some related results to the maximal family, which are the dual results of the above section.

Proposition 3.10. Let $P$ be a complete poset, $a \in P$. Then union of maximal families of $a$ are maximal families of $a$ as well.

Proof of this result is the dual of the minimal family.

IV. Molecular Posets

In this section we investigate some properties of Molecular Posets.

Definition 4.1. Let $P$ be a poset, then non-null -irreducible elements of $P$ is called a molecule, and $M$ denote set of all molecule of $P$. In the sequel, if $P$ is a CD poset, we prefer to call $P$ a molecular poset and write it in the form $P(M)$.

An element $a \in P$ is said to be $V$-irreducible if $a \leq b \land c$ implies that $a \leq b$ or $a \leq c$.

Lemma 4.2. In a poset $P$ the following two are equivalent

1. $a \in P$ is $V$-irreducible, $a$ is not a join of two different elements in $P$.
2. An element $a \in P$ is said to be $V$-irreducible if $a \leq b \land c$ implies that $a \leq b$ or $a \leq c$.

Proof of this equivalence is trivial.

Proposition 4.3. Let $P$ be a CD poset. Then each element of $P$ is a union of $V$-irreducible elements.

Proof. For $p \in P$, let $\Pi(p) = \{x \leq p \land x \leq \beta(x)\}$. Then $\Pi(p) \subseteq \beta\Pi(p)$ hence we only to prove that $\Pi(p) \subseteq \Pi(p)^0$.

Suppose that $d' = \Pi(p)^0 \supseteq p'$ is not true then there exists $b \in a_0(a)$ such that $p \not\leq b$. Let $I$ be an ideal defined as $a \in I \subseteq (b)$ and $\forall x \in P \setminus I$ there exists a minimal element $m \in P \setminus I$ such that $x \leq m$ then $a \in I \subseteq (b)$. Since $P$ is not in $(b)$ we have $p \in P \setminus I$. By Lemma 3.12 there is a minimal element $m \in P \setminus I$ satisfying $m \leq p$. $m$ is a $V$-irreducible element. Indeed if $m' \subseteq (x, y)^0$ and $m \not\subseteq x$, then $m' = \{x, y)^0 \land m, (x, y)^0 \} \neq m'$. Since $m$ is a minimal element in $P \setminus I$, it follows that $(m, x)^0$ not contained in $P \setminus I$, $(m, x)^0$ not contained in $P \setminus I$, and $m = \{x, y)^0 \land m, (x, y)^0 \} \neq m'$. Therefore $m$ is an ideal, contradicting the fact that $m \in P \setminus I$. This shows that $m$ is a $V$-irreducible, hence $m \in I(p)$, and so $m' \subseteq \Pi(p)^0 \supseteq d'$. But this implies that $m \in I$ because $I$ is lower set. This contradicts to the fact that $m \in P \setminus I$. Hence $d' = \Pi(p)^0 \supseteq p'$.
Lemma 4.4. Let $L$ be lattice and $a \in L$. By Proposition 4.3 if $\forall x \in \beta(a), [x]$ denotes the set of all $\mathcal{V}$-irreducible elements which are smaller than or equal to $x$, then $x = \text{sup}[x]$ if $\beta'(a) = \bigcup \{x / x \in \beta(a)\}$ then $\beta'(a)$ is a minimal family of $a$.

Proof. Let $[x] \subseteq \bigcup \{x / x \in \beta(a)\}$ is minimal family for $a$. First we show that $a = \text{sup}(\beta'(a))$. $x \in \beta(a)$ implies that $x \leq a$ (since $\beta(a)$ is lower set), therefore $\text{sup}(\beta'(a)) = \bigcup \{x / x \leq a\} = \{\text{all} \ \mathcal{V}-\text{irreducible elements below} \ a\} = a$ since $a$ is join of $\mathcal{V}$-irreducible elements.

We can prove the same result for poset in same way.

Lemma 4.5. Let $P$ be CD poset and $a \in P$. By Proposition 4.3 if $\forall x \in \beta(a), [x]$ denotes the set of all $\mathcal{V}$-irreducible elements which are smaller than or equal to $x$, then $x = \text{sup}[x]$ if $\beta'(a) = \bigcup \{x / x \in \beta(a)\}$ then $\beta'(a)$ is a minimal family of $a$.

Proof. Same as above.

Definition 4.6. Let $P$ be a complete lattice $a \in P, B \subset P$. $B$ is said to be a standard minimal family of $a$ if $B$ is a minimal family of $a$ and members of $B$ are $\mathcal{V}$-irreducible elements.

Theorem 4.7. Let $P$ be a complete poset. Then $P$ is a CD poset if and only if $\forall a \in P, a$ has a standard minimal family.

Proof. This theorem follows from above Proposition 4.3 and Lemma 4.5.

Definition 4.8. Let $P_1$ and $P_2$ molecular posets and $f : P_1 \to P_2$ a mapping $f$ will be called a generalized order-homomorphism or briefly, a GOH if

1. $f(0) = 0$
2. $f$ is union preserving
3. $f^{-1}$ is union preserving, where $\forall b \in P_2, f^{-1}(b) = \{a \in P_1 / f(a) \leq b\}^{\text{ul}}$

Theorem 4.9. Let $f : P_1 \to P_2$ be a GOH. Then

1. $f$ and $f^{-1}$ are order preserving
2. $f^{-1}(a) \geq a, \forall a \in P_1$
3. $f^{-1}(b) \leq b, \forall b \in P_2$
4. $f(a) \leq b$ if and only if $a \leq f^{-1}(b)$
5. $f(a) = \{b \in P_2 / f^{-1}(b) \geq a\}$
6. $f^{-1} : P_2 \to P_1$ is intersection preserving.

Proof. (1) Let $a, b \in P_1, a \leq b$ i.e. $a \lor b = b$ then $f(b) = f(a \lor b) = f(a) \lor f(b) \implies f(a) \leq f(b)$, since $f$ is join preserving. Similarly $f^{-1}(b) = f^{-1}(a \lor b) = f^{-1}(a) \lor f^{-1}(b)$ since $f^{-1}$ is join preserving, implies that $f^{-1}(a) \leq f^{-1}(b)$. Therefore $f$ and $f^{-1}$ are order preserving.

(2) By reflexivity $a \leq a$ and by (1) we have $f(a) \leq f(a)$. Now by definition of $f^{-1}$ we have $f^{-1}(f(a)) \subseteq \{a \in P_1 / f(a) \leq f(a)\}$ implies that $f^{-1}(f(a)) \geq a \forall a \in P_1$.

(3) $\{f^{-1}(b)\} = \{a \in P_1 / f(a) \leq b\}$ implies therefore $\{f^{-1}(b)\} = \{a \in P_1 / f(a) \leq b\}^{\text{ul}}$. Hence $f^{-1}(b) \leq b$.

(4) Let $a \in P_1, b \in P_2$ and $f(a) \leq b$, as $f^{-1}$ is order preserving therefore $f^{-1}(f(a)) \leq f^{-1}(b)$ but we have $a \geq f^{-1}(f(a)), \forall a \in P_1, a \leq f^{-1}(b)$.

(5) By (4) $f(a) \leq b$ if and only if $a \leq f^{-1}(b)$ i.e. $b$ is an upper bound for $f(a)$ in $P_2$. Therefore $f(a)^* = \text{maximal lower bound of such } b$'s such that $a \leq f^{-1}(b) = \{b \in P_2 / a \leq f^{-1}(b)\}^{\text{up}} \forall a \in P_1$.

(6) It is state forward.

Theorem 4.10. Let $f : P_1 \to P_2$ be a GOH

1. If $a$ is $\lor$-irreducible in $P_1$, then $f(a)$ is $\lor$-irreducible in $P_2$.
2. If $B$ is a minimal family of $a$ in $P_1$, then $f(B)$ is a minimal family of $f(a)$ in $P_2$.
3. If $B'$ is a standard minimal family of $a$ in $P_1$, then $f(B')$ is a standard minimal family of $f(B)$ in $P_2$.

Proof. (1) Let $a \in P_1$ be $\lor$-irreducible and $f(a) \leq b \lor c$ Then $a \leq f^{-1}(b \lor c) = f^{-1}(b) \lor f^{-1}(c)$. Since $a$ is $\lor$-irreducible we have $a \leq f^{-1}(b)$ or $a \leq f^{-1}(c)$ and hence $f(a) \leq f^{-1}(b) \lor f^{-1}(c) \leq f^{-1}(b) \lor f^{-1}(c)$. This proves that $f(a)$ is $\lor$-irreducible in $P_2$.

(2) Let $B$ be a minimal family of $a$ in $P_1$, then $f(B)^{\text{ul}} = f(B) \subseteq f(a)$ in $P_2$. Suppose that $C \subseteq P_2, C^{\text{ul}} \geq f(a)$ and $f(C)$. Then there exists $x \in B$ such that $f(x) = y$. Since $f^{-1}(C)^{\text{ul}} = f^{-1}(C) \supseteq f^{-1}(f(a)) \geq a$, by the definition of $B$ we know that there exists $z \in f^{-1}(C)$ such that $x \leq z$. Now $y = f(x) \leq f(z)$ in $C$. This proves that $f(B)$ is a minimal family of $f(a)$ in $P_2$.

(3) This property follows directly from (1) and (2).
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