STABILITY AND NON-STABILITY OF FUNCTIONAL EQUATIONS IN MATRIX NORMED SPACES

R. Murali, V. Vithya

Abstract: In this paper, we attempt to find some stability results of hexadecic functional equations in matrix normed spaces with the help of fixed point and direct methods and we also investigate the stability of quindecic functional equation in matrix normed spaces by using the direct method. Also, we give an example for non-stability cases for these functional equations.

Index Terms - Stability, fixed point, hexadecic functional equation, quindecic functional equation, direct method, matrix normed spaces.

I. INTRODUCTION

Stability problem of a functional equation was first posed by Ulam [24] and that was partially answered by Hyers [5] and then generalized by Aoki [1] and Rassias [18] for additive mappings and linear mappings, respectively. In 1994, a generalization of Rassias's theorem was obtained by Găvruta [4], who replaced $\varepsilon\|x\|^p + \|y\|^p$ by a general control function $\phi(x, y)$. This idea is known as generalized Hyers-Ulam-Rassias stability. In 1996, G. Isac and Th.M. Rassias [6] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. This terminology may also be applied to the cases of other functional equations such as additive, quadratic, cubic and quartic [9], quintic and sextic [25], septic and octic [23], nonic [19], decic [2], undecic [21], duodecic [20], tredecic [13], quattuordecic [22] functional equations have been investigated by a number of authors with more general domains and co-domains. Recently, C. Park et al. [7] researched the the Ulam stability of Cauchy additive and quadratic functional equations in matrix normed spaces.

Quite recently, R. Murali et al. [12] introduced the following hexadecic functional equation

\[
\begin{align*}
    f(x + y) & = 16f(x + y) + 16f(x + 6y) - 560f(x + 5y) + 1820f(x + 4y) - 4368f(x + 3y) + 8008f(x + 2y) - 11440f(x + y) + 12870f(x) - 11440f(x - y) + 8008f(x - 2y) - 4368f(x - 3y) + 1820f(x - 4y) - 560f(x - 5y) + 120f(x - 6y) - 16f(x - 7y) + f(x - 8y) = 20922789890000f(y).
\end{align*}
\]

in muti-banach spaces with the help of fixed point method.

and quindecic functional equation [11] as follows

\[
\begin{align*}
    f(x + y) & = 15f(x + y) + 105f(x + 6y) - 1365f(x + 5y) + 3003f(x + 4y) - 3003f(x + 3y) + 5005f(x + 2y) - 6435f(x + y) + 6435f(x) - 1365f(x - y) + 3003f(x - 2y) - 1365f(x - 3y) + 455f(x - 4y) - 105f(x - 5y) + 15f(x - 6y) - f(x - 7y) = 15!f(y).
\end{align*}
\]

Where 15! = 1307674368000.

In this paper, we study the generalized Hyers-Ulam-Rassias, Hyers-Ulam-Rassias, Ulam-Gavruta-Rassias and J.M. Rassias stability results for the functional equation (1) in matrix normed spaces with the help of fixed point method and also prove the stability results for the above functional equations (1) and (1A) in matrix normed spaces by using direct method.

For a mapping $f: X \rightarrow Y$, define $Gf: X^2 \rightarrow Y$ by

\[
\begin{align*}
    Gf(b, c) & = f(b + 8c) - 16f(b + 7c) + 120f(b + 6c) - 560f(b + 5c) + 1820f(b + 4c) - 4368f(b + 3c) + 8008f(b + 2c) - 11440f(b + c) + 12870f(b) - 11440f(b - c) + 8008f(b - 2c) - 4368f(b - 3c) + 1820f(b - 4c) - 560f(b - 5c) + 120f(b - 6c) - 16f(b - 7c) + f(b - 8c) = 20922789890000f(c).
\end{align*}
\]

and $Gf_n: M_n(X) \rightarrow M_n(Y)$ by

\[
\begin{align*}
    Gf([x_{rs}, y_{rs}]) & = f([x_{rs} + 8y_{rs}]) - 16f([x_{rs} + 7y_{rs}]) + 120f([x_{rs} + 6y_{rs}]) - 560f([x_{rs} + 5y_{rs}]) + 1820f([x_{rs} + 4y_{rs}]) - 4368f([x_{rs} + 3y_{rs}]) + 8008f([x_{rs} + 2y_{rs}]) - 11440f([x_{rs} + y_{rs}]) + 12870f([x_{rs}]) - 11440f([x_{rs} - y_{rs}]) + 8008f([x_{rs} - 2y_{rs}]) - 4368f([x_{rs} - 3y_{rs}]) + 1820f([x_{rs} - 4y_{rs}]) - 560f([x_{rs} - 5y_{rs}]) + f([x_{rs} - 6y_{rs}]) + 120f([x_{rs} - 7y_{rs}]) - 16f([x_{rs} - 8y_{rs}]) = 20922789890000f([y_{rs}]).
\end{align*}
\]

for all $b, c \in X$ and all $x = [x_{rs}], y = [y_{rs}] \in M_n(X)$. 
II. STABILITY OF HEXADEC FUNCTIONAL EQUATION (1) : FIXED POINT METHOD AND DIRECT METHOD

In this section, we investigate Generalized Hyers-Ulam-Rassias stability, Hyers-Ulam stability, Ulam -Gavruta- Rassias stability and J. M. Rassias stability for the functional equation (1) in matrix normed spaces by using fixed point method and the direct method.

Theorem 1 Let \( l = \pm 1 \) be fixed and \( \zeta : X^2 \to [0, \infty) \) be a function such that there exists a \( \delta < 1 \) with

\[
\zeta(b, c) \leq 2^{16l} \delta \frac{b}{2^{16l + 1}} \quad \forall b, c \in X. \tag{2}
\]

Let \( f : X \to Y \) be a mapping such that

\[
\|Gf_n([x_{rs}], [y_{rs}])\| \leq \sum_{\tau, s=1}^{\infty} \zeta(x_{rs}, y_{rs}) \quad \forall x = [x_{rs}], y = [y_{rs}] \in M_n(X). \tag{3}
\]

Then there exists a unique hexadecic mapping \( H : X \to Y \) such that

\[
\|f_n([x_{rs}]) - H_n([x_{rs}])\| \leq \sum_{\tau, s=1}^{\infty} \frac{1 - \delta}{2^{16l(1-\delta)}} \zeta(x_{rs}) \quad \forall x = [x_{rs}] \in M_n(X), \tag{4}
\]

where \( \zeta^*(x_{rs}) = \frac{2}{16l} \left[ \frac{1}{2} \zeta(0, 2x_{rs}) + \zeta(8x_{rs}, x_{rs}) + 16\zeta(7x_{rs}, x_{rs}) + 120\zeta(6x_{rs}, x_{rs}) + 560\zeta(5x_{rs}, x_{rs}) + 1802\zeta(4x_{rs}, x_{rs}) + 4368\zeta(3x_{rs}, x_{rs}) + 8008\zeta(2x_{rs}, x_{rs}) + 11440\zeta(x_{rs}, x_{rs}) + 6435\zeta(0, x_{rs}) \right]. \]

Proof. For the cases \( l = 1 \) and \( l = -1 \), substituting \( n = 1 \) in (3), we obtain

\[
\|Gf(b, c)\| \leq \zeta(b, c) \tag{5}
\]

Let \( b = 0 \) and replacing \( c \) by \( 2b \) in (5), one gets

\[
\|f(16u) - 16f(14u) + 120f(12u) - 560f(10u) + 1820f(8u) - 4368f(6u) + 8008f(4u) - 1046139495000f(2u)\| \leq \frac{1}{2} \zeta(0, b) \tag{6}
\]

for all \( b \in X \). Putting \( b = 8b \) and \( c = b \) in (5), we arrive at

\[
\|f(16u) - 16f(15u) + 120f(14u) - 560f(13u) + 1820f(12u) - 4368f(11u) + 8008f(10u) - 11440f(9u) + 12870f(8u) - 11440f(7u) + 8008f(6u) - 4368f(5u) + 1820f(4u) - 560f(3u) + 120f(2u) - 16f(u)\| \leq \zeta(8b, b) \tag{7}
\]

for all \( b \in X \). Combining (6) and (7), we get

\[
\|16f(15u) - 136f(14u) + 560f(13u) - 1700f(12u) + 4368f(11u) - 8568f(10u) + 11440f(9u) - 11050f(8u) + 11440f(7u) - 12376f(6u) + 4368f(5u) + 6188f(4u) + 560f(3u) - 1046139495000f(2u) + 16f(u)\| \leq \frac{1}{2} \zeta(0, 2b) + \zeta(8b, b) \tag{8}
\]

Substituting \( b = 7b \) and \( c = b \) in (5), further multiplying the resulting inequality by 16, and combining the obtained result to (8), we get

\[
\|120f(14u) - 1360f(13u) + 7260f(12u) - 24752f(11u) + 61320f(10u) - 11688f(9u) + 171990f(8u) - 194480f(7u) + 170664f(6u) - 123760f(5u) + 76076f(4u) - 28560f(3u) - 1046139495000f(2u) + 16f(17f(u))\| \leq \frac{1}{2} \zeta(0, 2b) + \zeta(8b, b) + 16\zeta(7b, b) \tag{9}
\]

for all \( b \in X \). Replacing \( b \) by \( 6b \) and \( c = b \) in (5), further multiplying the resulting inequality by 120, and combining the obtained result to (9), we get

\[
\|560f(13u) - 7140f(12u) + 42448f(11u) - 157080f(10u) + 407472f(9u) - 788970f(8u) + 1178320f(7u) - 1373736f(6u) + 1240940f(5u) - 884884f(4u) + 495600f(3u) - 1046139515000f(2u) + 16f(137f(u))\| \leq \frac{1}{2} \zeta(0, 2b) + \zeta(8b, b) + 16\zeta(7b, b) + 120\zeta(6b, b) \tag{10}
\]

for all \( b \in X \). If we take \( b = 5b \) and \( c = b \) in (5), further multiplying the resulting inequality by 560 and combining the obtained result to (10), we get

\[
\|1820f(12u) - 24752f(11u) + 156520f(10u) - 611728f(9u) + 1657110f(8u) - 3306160f(7u) + 5032664f(6u) - 5958160f(5u) + 5521516f(4u) - 3989440f(3u) - 10461392700000f(2u) + 16f(697f(u))\| \leq \frac{1}{2} \zeta(0, 2b) + \zeta(8b, b) + 16\zeta(7b, b) + 120\zeta(6b, b) \tag{11}
\]
for all \( b \in X \). Putting \( b = 4b \) and \( c = b \) in (5), further multiplying the resulting inequality by 1820, and combining the obtained result to (11), we get

\[
\leq \frac{1}{2} \zeta(0,2b) + \zeta(8b,b) + 16\zeta(7b,b) + 120\zeta(6b,b) + 560\zeta(5b,b)
\]

(11)

for all \( b \in X \). Replacing \( b = 3b \) and \( c = b \) in (5), further multiplying the resulting inequality by 4368, and combining the obtained result from (12), we get

\[
\|8008f(10u) - 116688f(9u) + 790790f(8u) - 3306160f(7u) + 953752f(6u) - 20120672f(5u) + 32136104f(4u) - 39879840f(3u) - 10461355080000f(2u) + 16!(8885)f(u)\|
\leq \frac{1}{2} \zeta(0,2b) + \zeta(8b,b) + 16\zeta(7b,b) + 120\zeta(6b,b) + 560\zeta(5b,b) + 1820\zeta(4b,b) + 4368\zeta(3b,b) + 8008\zeta(2b,b)
\]

(13)

for all \( b \in X \). Letting \( b = 2b \) and \( c = b \) in (5), further multiplying the resulting inequality by 8008, and combining the obtained result to (13), we get

\[
\|11440f(9u) - 170170f(8u) + 1178320f(7u) - 5045040f(6u) + 14986400f(5u) - 32959290f(4u) + 56216160f(3u) - 10461472720000f(2u) + 16!(14893)f(u)\|
\leq \frac{1}{2} \zeta(0,2b) + \zeta(8b,b) + 16\zeta(7b,b) + 120\zeta(6b,b) + 560\zeta(5b,b)
+ 1820\zeta(4b,b) + 4368\zeta(3b,b) + 8008\zeta(2b,b) + 11440\zeta(0,b,b)
\]

(14)

for all \( b \in X \). Taking \( b = c = b \) in (5), further multiplying the resulting inequality by 11440, and combining the obtained result to (14), we get

\[
\|12870f(8u) - 205920f(7u) + 1544400f(6u) - 7207200f(5u) + 23423400f(4u) - 56216160f(3u) - 10461291870000f(2u) + 16!(26333)f(u)\|
\leq \frac{1}{2} \zeta(0,2b) + \zeta(8b,b) + 16\zeta(7b,b) + 120\zeta(6b,b) + 560\zeta(5b,b)
+ 1820\zeta(4b,b) + 4368\zeta(3b,b) + 8008\zeta(2b,b) + 11440\zeta(0,b,b)
\]

(15)

for all \( b \in X \). Replacing \( b = 0 \) and \( c = b \) in (5), further multiplying the resulting inequality by 12870, and combining the obtained result to (15), we get

\[
\|f(2b) + 65536f(b)\| \leq \frac{1}{2} \frac{1}{16}(\frac{1}{2}\zeta(0,2b) + \zeta(8b,b) + 16\zeta(7b,b) + 120\zeta(6b,b) + 560\zeta(5b,b)
+ 1820\zeta(4b,b) + 4368\zeta(3b,b) + 8008\zeta(2b,b) + 11440\zeta(0,b,b))
\]

(16)

for all \( b \in X \). So

\[
\left\|f(b) - \frac{1}{2^{16}}f(2b)\right\| \leq \frac{\delta}{2^{16}}\zeta^*(b)
\]

(17)

for all \( b \in X \). Set \( \mathcal{N} = \{ f : X \to Y \} \) and the generalized metric \( \rho \) on \( \mathcal{N} \) as follows:

\[\rho(f, g) = \inf\{\mu \in \mathbb{R}_+: \|f(0) - f(b)\| \leq \mu\zeta^*(c), \forall b \in X\},\]

Claim:1 It is easy to verify that \((E, \rho)\) is a complete generalized metric.(see [8]).

Claim:2 \( \mathcal{T} \) be a strictly contractive mapping with a Lipschitz constant is less than 1.

Consider the mapping \( \mathcal{T} : \mathcal{N} \to \mathcal{N} \) defined by \( \mathcal{T}f(b) = \frac{1}{2^{16}}f(2b) \) \( \forall f \in \mathcal{N}, b \in X \).

Hence \( \mathcal{T} \) is a contractive mapping with constant \( \tau < 1 \). From (17), we can get \( \rho(f, \mathcal{T}f) \leq \frac{\delta^{1-\tau}}{2^{16}}\).

Together Claim 1 and 2 (Theorem 2.2 in [3]), then there exists a mapping \( \mathcal{H} : X \to Y \) which satisfying:

1. \( \mathcal{H} \) is a unique fixed point of \( \mathcal{T} \), which is satisfied \( \mathcal{H}(2b) = 2\mathcal{H}(b) \) \( \forall b \in X \).
2. \( \rho(\mathcal{T}^k f, \mathcal{H}) \to 0 \) as \( k \to \infty \). This implies that \( \lim_{k \to \infty} \frac{1}{2^{16}}f(2^k b) = \mathcal{H}(b) \) \( \forall b \in X \).
3. \( \rho(f, \mathcal{H}) \leq \frac{1}{1-\delta}\rho(f, \mathcal{T}f) \), which implies the inequality

\[\|f(b) - \mathcal{H}(b)\| \leq \frac{\delta}{2^{16}(1-\delta)}\zeta^*(c) \quad \forall b \in X.\]

(18)

It follows from (2) and (3), \( \mathcal{H} \) is hexadecic mapping. By Lemma 2.1 in [9] and (18), we can get (4) Thus \( \mathcal{H} : X \to Y \) is a unique hexadecic mapping satisfying (4).

Corollary 1 Let \( l = \pm 1 \) be fixed and let \( t, y \) be non-negative real numbers with \( t \neq 16 \). Let \( h : X \to Y \) be a mapping such that

\[\|Gf_n([x_{r_1}], [y_{r_2}])\|_n \leq \sum_{r=1}^n \gamma(\|x_{r_1}\| + \|y_{r_2}\|) \quad \forall x = [x_{r_1}], y = [y_{r_2}] \in M_n(X)\]

(19)
Then there exists a unique hexadecic mapping \( \mathcal{F}: X \to Y \) such that
\[
\|f_n([x_{rs}]) - \mathcal{F}_n([x_{rs}])\|_n \leq \sum_{r,s=1}^{n} \gamma_{0} \|x_{rs}\|_f \quad \forall x = [x_{rs}] \in M_n(X),
\]
where \( \gamma_{0} = \frac{2\gamma_{16}}{16} [44208 + 8008(2^3) + 4368(3^4) + 1820(4^4) + 560(5^4) + 120(6^4) + 16(7^4) + 8^4] \).

The proof is identical to the proof of Theorem 1.

**Corollary 2** Let \( l = \pm 1 \) be fixed and let \( t, \gamma \) be non-negative real numbers with \( t \neq 16 \). Let \( h: X \to Y \) be a mapping such that
\[
\|g_{f_n}([x_{rs}],[y_{rs}])\|_n \leq \sum_{r,s=1}^{n} \gamma (\|x_{rs}\|_d \cdot \|y_{rs}\|_f^e) \forall x = [x_{rs}], y = [y_{rs}] \in M_n(X)
\]
Then there exists a unique hexadecic mapping \( \mathcal{F}: X \to Y \) such that
\[
\|f_n([x_{rs}]) - \mathcal{F}_n([x_{rs}])\|_n \leq \sum_{r,s=1}^{n} \gamma_{0} \|x_{rs}\|_f \quad \forall x = [x_{rs}] \in M_n(X),
\]
where \( \gamma_{0} = \frac{2\gamma_{16}}{16} [11440 + 8008(2^4) + 4368(3^4) + 1820(4^4) + 560(5^4) + 120(6^4) + 16(7^4) + 8^4] \).

The proof is identical to the proof of Theorem 1.

**Corollary 3** Let \( l = \pm 1 \) be fixed and let \( t, \gamma \) be non-negative real numbers with \( t \neq 16 \). Let \( h: X \to Y \) be a mapping such that
\[
\|g_{f_n}([x_{rs}],[y_{rs}])\|_n \leq \sum_{r,s=1}^{n} \gamma (\|x_{rs}\|_d \cdot \|y_{rs}\|_f^e + \|x_{rs}\|_f^{d+e} + \|y_{rs}\|_f^{d+e}) \forall x = [x_{rs}], y = [y_{rs}] \in M_n(X)
\]
Then there exists a unique hexadecic mapping \( \mathcal{F}: X \to Y \) such that
\[
\|f_n([x_{rs}]) - \mathcal{F}_n([x_{rs}])\|_n \leq \sum_{r,s=1}^{n} \gamma_{0} \|x_{rs}\|_f \quad \forall x = [x_{rs}] \in M_n(X),
\]
where \( \gamma_{0} = \frac{2\gamma_{16}}{16} [55648 + 8008(2^4) + 8008(2^4) + 4368(3^4 + 3^4) + 1820(4^4 + 4^d) + 560(5^4 + 5^d) + 120(6^4 + 6^d) + 16(7^4 + 7^d) + 8^4 + 8^4] \).

The proof is similar to the proof of Theorem 1.

**Theorem 2** Let \( l = \pm 1 \) be fixed and \( \zeta: X^2 \to [0,\infty) \) be a function such that
\[
\sum_{m=0}^{\infty} \frac{\zeta(2^mb,2^mc)}{2^{16ml}} < +\infty
\]
and
\[
\lim_{m \to \infty} \frac{\zeta(2^mlb,2^mlc)}{2^{27ml}} = 0
\]
for all \( b,c \in X \). Let \( f: X \to Y \) be a mapping such that (4). Then there exists a unique hexadecic mapping \( \mathcal{H}: X \to Y \) such that
\[
\|f_n([x_{rs}]) - \mathcal{H}_n([x_{rs}])\|_n \leq \sum_{r,s=1}^{n} \gamma_{0} \|x_{rs}\|_f \quad \forall x = [x_{rs}] \in M_n(X),
\]
where
\[
\gamma_{0} = \frac{2\gamma_{16}}{16} [0.22^{ml}x_{rs} + \zeta(2^{ml}x_{rs},2^{ml}x_{rs}) + 16\zeta(72^{ml}x_{rs},2^{ml}x_{rs}) + 120\zeta(62^{ml}x_{rs},2^{ml}x_{rs}) + 560\zeta(52^{ml}x_{rs},2^{ml}x_{rs}) + 1820\zeta(42^{ml}x_{rs},2^{ml}x_{rs}) + 4368\zeta(32^{ml}x_{rs},2^{ml}x_{rs}) + 8008\zeta(22^{ml}x_{rs},2^{ml}x_{rs}) + 11440\zeta(2^{ml}x_{rs},2^{ml}x_{rs}) + 6435\zeta(0.2^{ml}x_{rs})]
\]
Proof. From (16)
\[
\|f(b) - \frac{1}{2^{16}} f(2b)\| \leq \frac{1}{2^{16}} \zeta^*(b)
\]
for all \( b \in X \). Now replacing \( b \) by \( 2b \) and dividing \( 2^{16} \) in (26), we have
for all $b \in X$. From (26) and (27), we obtain

$$
\left\| f \left( \frac{2^l b}{2^{32}} \right) - f \left( \frac{2^l b}{2^{16}} \right) \right\| \leq \frac{1}{2^{16}} \zeta^* \left( \frac{2^l b}{2^{16}} \right)
$$

(27)

For case $l = 1$ and $l = -1$, Generalizing for any positive integer $q$, we get

$$
\left\| f \left( \frac{2^q b}{2^{16q}} \right) - f \left( \frac{2^q b}{2^{16}} \right) \right\| \leq \frac{1}{2^{16}} \sum_{m=1}^{q-1} \zeta^* \left( \frac{2^m b}{2^{16}} \right)
$$

(28)

To prove the convergence of the sequence $\frac{f (2^q b)}{2^{16q}}$, we replace $b$ by $2^m b$ in (29) and divide the resultant by $2^{16m}$, for any $q, m > 0$, we get

$$
\left\| f \left( \frac{2^{q+m} b}{2^{16(q+m)}} \right) - f \left( \frac{2^m b}{2^{16m}} \right) \right\| = \frac{1}{2^{16m}} \left\| f \left( 2^{q} \frac{2^m b}{2^{16m}} \right) - f \left( \frac{2^m b}{2^{16m}} \right) \right\| \leq \frac{1}{2^{16m}} \sum_{i=0}^{q-1} \zeta^* \left( \frac{2^i 2^m b}{2^{16}} \right)
$$

$$
= \frac{1}{2^{16}} \sum_{i=0}^{q-1} \zeta^* \left( \frac{2^{i+m} b}{2^{16(l+m)}} \right) \to 0 \quad \text{as} \quad m \to \infty
$$

(29)

for all $b \in X$. Thus, it follows that the sequence $\frac{f (2^q b)}{2^{16q}}$ is Cauchy in $Y$ and so it converges. Therefore, the mapping $H : X \to Y$ is defined by

$$
H(b) = \lim_{q \to \infty} \frac{f (2^q b)}{2^{16q}}
$$

(30)

is well defined for all $b \in X$. On the other hand it follows that for $f \in \Gamma_{2^l}(b, b)$ that

$$
\| H(b, c) - H(b, c) \| \leq \frac{1}{2^{16l}} \left\| Gf \left( \frac{2^l b}{2^{16l}} \right) \right\| \leq \frac{1}{2^{16l}} \zeta^* \left( \frac{2^l b}{2^{16l}} \right) = 0
$$

for all $b, c \in X$. So, the mapping $H$ is hexadecic. Hence $H$ satisfies (1). To prove that $H$ is unique, we assume that there is $H'$ as another hexadecic mapping satisfies (1).

$$
\| H(b) - H'(b) \| = \frac{1}{2^{16l}} \left\| H(b) - H'(b) \right\| \leq \frac{1}{2^{16l}} \left\| H(b) - f \left( \frac{2^q a}{2^{16l}} \right) \right\| + \left\| f \left( \frac{2^q a}{2^{16l}} \right) - H'(b) \right\| \leq \frac{1}{2^{16l}} \sum_{i=0}^{q-1} \zeta^* \left( \frac{2^{i(q+q)} b}{2^{16(l+i+q)}} \right) \to 0 \quad \text{as} \quad q \to \infty
$$

(31)

By Lemma 2.1 in [9] and (29), we can get (25). Thus $H : X \to Y$ is a unique hexadecic mapping satisfying (25).

### III. Stability of Quindecic Functional Equation (1A): Direct Method

In this section, we investigate Generalized Hyers-Ulam-Rassias stability, Hyers-Ulam stability, Ulam -Gavruta- Rassias stability and J. M. Rassias stability for the functional equation (1A) in matrix normed spaces by using the direct method.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n (X^2) \to M_n (Y)$ by,

$$
Df(b, c) = f(b + 8c) - 15 f(b + 7c) + 105 f(b + 6c) - 455 f(b + 5c) + 1365 f(b + 4c) - 3003 f(b + 3c) + 5005 f(b + 2c) - 6435 f(b + c) + 6435 f(b)
$$

$$
- 5005 f(b - c) + 3003 f(b - 2c) - 1365 f(b - 3c) + 455 f(b - 4c) - 105 f(b - 5c) + 15 f(b - 6c) - f(b - 7c) - 15 f(y)
$$

$$
Df_n(x_{ir}, y_{ir}) = f_n (x_{ir} + 8 y_{ir}) - 15 f_n (x_{ir} + 7 y_{ir}) + 105 f_n (x_{ir} + 6 y_{ir}) + 1365 f_n (x_{ir} + 5 y_{ir}) - 3003 f_n (x_{ir} + 4 y_{ir}) - 5005 f_n (x_{ir} + 3 y_{ir}) + 6435 f_n (x_{ir} + 2 y_{ir}) - 6435 f_n (x_{ir} + y_{ir}) - 3003 f_n (x_{ir} - 2 y_{ir}) + 1365 f_n (x_{ir} - 3 y_{ir}) - 455 f_n (x_{ir} - 4 y_{ir}) - 105 f_n (x_{ir} - 5 y_{ir}) + 15 f_n (x_{ir} - 6 y_{ir}) - f_n (x_{ir} - 7 y_{ir}) - 15 f_n (y_{ir})
$$

$$
Df_n(x_{ir}, y_{ir}) + 15 f_n (x_{ir} + 7 y_{ir}) - 105 f_n (x_{ir} + 6 y_{ir}) + 1365 f_n (x_{ir} + 5 y_{ir}) - 3003 f_n (x_{ir} + 4 y_{ir}) - 5005 f_n (x_{ir} + 3 y_{ir}) + 6435 f_n (x_{ir} + 2 y_{ir}) - 6435 f_n (x_{ir} + y_{ir}) - 3003 f_n (x_{ir} - 2 y_{ir}) + 1365 f_n (x_{ir} - 3 y_{ir}) - 455 f_n (x_{ir} - 4 y_{ir}) - 105 f_n (x_{ir} - 5 y_{ir}) + 15 f_n (x_{ir} - 6 y_{ir}) - f_n (x_{ir} - 7 y_{ir}) - 15 f_n (y_{ir})
$$

$$
Df_n(x_{ir}, y_{ir}) + 15 f_n (x_{ir} + 7 y_{ir}) - 105 f_n (x_{ir} + 6 y_{ir}) + 1365 f_n (x_{ir} + 5 y_{ir}) - 3003 f_n (x_{ir} + 4 y_{ir}) - 5005 f_n (x_{ir} + 3 y_{ir}) + 6435 f_n (x_{ir} + 2 y_{ir}) - 6435 f_n (x_{ir} + y_{ir}) - 3003 f_n (x_{ir} - 2 y_{ir}) + 1365 f_n (x_{ir} - 3 y_{ir}) - 455 f_n (x_{ir} - 4 y_{ir}) - 105 f_n (x_{ir} - 5 y_{ir}) + 15 f_n (x_{ir} - 6 y_{ir}) - f_n (x_{ir} - 7 y_{ir}) - 15 f_n (y_{ir})
$$
for all \( b, c \in X \) and all \( x = [x_{rs}], y = [y_{rs}] \in M_n(X) \).

**Theorem 3** Let \( l = \pm 1 \) be fixed and \( \zeta : X^2 \to [0, \infty) \) be a function such that

\[
\sum_{m=0}^{\infty} \frac{\zeta(2^{ml}b, 2^{ml}c)}{2^{15ml}} < \infty
\]

and

\[
\lim_{m \to \infty} \frac{\zeta(2^{ml}b, 2^{ml}c)}{2^{15ml}} = 0
\]

for all \( b, c \in X \). Let \( f : X \to Y \) be a mapping satisfies

\[
\|Df_n([x_{rs}], [y_{rs}])\| \leq \sum_{r,s=1}^{\infty} \zeta(x_{rs}, y_{rs}) \forall x = [x_{rs}], y = [y_{rs}] \in M_n(X).
\]

Then there exists a unique quindecimal mapping \( Q : X \to Y \) such that

\[
\|f_n([x_{rs}]) - Q_n([x_{rs}])\|_{n} \leq \sum_{r,s=1}^{\infty} \frac{1}{2^{15}} \left( \sum_{m=\lceil l/2 \rceil}^{\infty} \frac{\zeta^*(2^{ml}x_{rs})}{2^{15ml}} \right)
\]

for all \( x = [x_{rs}] \in M_n(X) \), where

\[
\zeta^*(2^{ml}x_{rs}) = \frac{1}{15!} [\zeta(0, 2^{2ml}x_{rs}) + \zeta(2^{ml}x_{rs}, 2^{ml}x_{rs}) + 15\zeta(7.2^{ml}x_{rs}, 2^{ml}x_{rs}) + 106\zeta(6.2^{ml}x_{rs}, 2^{ml}x_{rs})
\]

\[
+ 470\zeta(5.2^{ml}x_{rs}, 2^{ml}x_{rs}) + 1470\zeta(4.2^{ml}x_{rs}, 2^{ml}x_{rs}) + 3458\zeta(3.2^{ml}x_{rs}, 2^{ml}x_{rs}) + 6370\zeta(2.2^{ml}x_{rs}, 2^{ml}x_{rs}) + 9438\zeta(2^{ml}x_{rs}, 2^{ml}x_{rs}) + 11440\zeta(0.2^{ml}x_{rs})]
\]

**Proof.** Put \( n = 1 \) in (32), we get

\[
\|Df(b, c)\| \leq \zeta(b, c)
\]

for all \( b, c \in X \). By the same reasoning as in [11], there is an inequality

\[
\left\| f(b) - \frac{1}{2^{15}} f(2b) \right\| \leq \frac{1}{2^{15}} \zeta^*(b)
\]

for all \( b \in X \). Now replacing \( b \) by \( 2b \) and dividing \( 2^{15} \) in (35), we have

\[
\left\| f(2^{2b}) - \frac{1}{2^{15}} f(2b) \right\| \leq \frac{1}{2^{10}} \zeta^*(2b)
\]

for all \( b \in X \). From (35) and (36), we obtain

\[
\left\| f(2^{2b}) - \frac{1}{2^{15}} f(2b) \right\| \leq \frac{1}{2^{15}} [\zeta^*(b) + \frac{1}{2^{15}} \zeta^*(2b)]
\]

For case \( l = 1 \) and \( l = -1 \), Generalizing for any positive integer \( q \) , we get

\[
\left\| f(2^{2b}) - \frac{1}{2^{15}} f(2b) \right\| \leq \frac{1}{2^{15}} \sum_{m=\lceil l/2 \rceil}^{\infty} \frac{\zeta^*(2^{ml}b)}{2^{15ml}}
\]

To prove the convergence of the sequence \( \frac{f(2^{q}b)}{2^{15q}} \), we replace \( b \) by \( 2^{ml}b \) in (38) and divide the resultant by \( 2^{15ml} \), for any \( q, m > 0 \), we get

\[
\left\| f(\frac{2^{q+m}b}{2^{15q}}) - \frac{1}{2^{15}} f(\frac{2^{q}b}{2^{15}
\]

\[
= \frac{1}{2^{15}} \frac{1}{2^{15}} \frac{1}{2^{15}} \sum_{m=\lceil l/2 \rceil}^{\infty} \frac{\zeta^*(2^{2^{ml}b})}{2^{15}} \to 0 \quad \text{as} \quad m \to \infty
\]

for all \( b \in X \). Thus, it follows that the sequence \( \frac{f(2^{q}b)}{2^{15q}} \) is Cauchy in \( Y \) and so it converges. Therefore, the mapping \( Q : X \to Y \) is defined by
is well defined for all \( b \in X \). On the other hand it follows from (31), (6) and (39) that
\[
\|Q(b, c)\| = \lim_{k \to \infty} \frac{1}{2^{15k}} \|DF(2^{15k} b, 2^{15k} c)\| \leq \frac{1}{2^{15k}} \zeta(2^{15k} b, 2^{15k} c) = 0
\]
for all \( b, c \in X \). So, the mapping \( Q \) is quindecic. Hence \( Q \) satisfies (1A). To prove that \( Q \) is unique, we assume that there is another quindecic mapping \( Q' \). Then there exists a unique quindecic mapping 
\[
Q(b) = \lim_{q \to \infty} \frac{f(2^{15q} b)}{2^{15q}}
\]
(39)

Corollary 4 Let \( l = \pm 1 \) be fixed and let \( t, \omega \) be non-negative real numbers with \( t \neq 15 \). Let \( f : X \to Y \) be a mapping such that
\[
\|DF_n([x_{rs}], [y_{rs}])\|_n \leq \sum_{r,s=1}^{n} \omega (\|x_{rs}\|^t + \|y_{rs}\|^t) \quad \forall x = [x_{rs}], y = [y_{rs}] \in M_n(X)
\]
Then there exists a unique quindecic mapping \( Q : X \to Y \) such that
\[
\|f_n([x_{rs}]) - Q_n([x_{rs}])\|_n \leq \sum_{i,j=1}^{n} \frac{\alpha_0}{2^{15t - 2^i}} \|x_{rs}\|^t
\]
for all \( x = [x_{rs}] \in M_n(X) \), where
\[
\alpha_0 = \frac{\omega}{15!} [42206 + 6371(2') + 3458(3') + 1470(4') + 470(5') + 106(6') + 15(7') + 8']
\]
Proof. The proof is similar to that of Theorem 3 by taking \( \zeta(b, c) \) as \( \omega(\|b\|^t + \|c\|^t) \) for all \( b, c \in X \).

Corollary 5 Let \( l = \pm 1 \) be fixed and let \( t, \omega \) be non-negative real numbers with \( t = v + w \neq 15 \). Let \( f : X \to Y \) be a mapping satisfies
\[
\|DF_n([x_{rs}], [y_{rs}])\|_n \leq \sum_{r,s=1}^{n} \omega (\|x_{rs}\|^v \|y_{rs}\|^w) \quad \forall x = [x_{rs}], y = [y_{rs}] \in M_n(X)
\]
Then there exists a unique quindecic mapping \( Q : X \to Y \) such that
\[
\|f_n([x_{rs}]) - Q_n([x_{rs}])\|_n \leq \sum_{i,j=1}^{n} \frac{\alpha_0}{2^{15w - 2^i}} \|x_{rs}\|^w
\]
for all \( x = [x_{rs}] \in M_n(X) \), where
\[
\alpha_0 = \frac{\omega}{15!} [9438 + 3458(2') + 1470(3') + 470(4') + 106(5') + 15(6') + 8']
\]
Proof. The proof is similar to that of Theorem 3 by taking \( \zeta(b, c) \) as \( \omega(\|b\|^v + \|c\|^w) \) for all \( b, c \in X \).

Corollary 6 Let \( l = \pm 1 \) be fixed and let \( t, \omega \) be non-negative real numbers with \( t = v + w \neq 15 \) Let \( f : X \to Y \) be a mapping satisfies
\[
\|DF_n([x_{rs}], [y_{rs}])\|_n \leq \sum_{r,s=1}^{n} \omega (\|x_{rs}\|^v \|y_{rs}\|^w + \|x_{rs}\|^t + \|y_{rs}\|^t) \quad \forall x = [x_{rs}], y = [y_{rs}] \in M_n(X)
\]
Then there exists a unique quindecic mapping \( Q : X \to Y \) such that
\[ \| f_n(x_n) - Q_n(x_n) \| \leq \sum_{i=1}^{n} \left\| \frac{\alpha_0}{2^{15} - 2^i} \right\| \| x_i \| \]

for all \( x = [x_n] \in M_n(X) \), where

\[ \alpha_0 = \frac{\alpha_0}{15} \left( 51644 + 6371(2^r) + 6370(2^r)3458(3^r + 3^r) + 1470(4^r + 4^r) + 470(5^r + 5^r) + 106(6^r + 6^r) + 15(7^r + 7^r) + 8^r + 8^r \right) \]

**Proof.** The proof is similar to that of Theorem 3 by taking \( \zeta(b, c) = \alpha(\|b\|^{16} + \|c\|^{16}) \) for all \( b, c \in X \).

**IV. EXAMPLES FOR NON-STABILITY**

The following examples illustrate that the functional equations (1) and (1A) is not stable for \( t = 16 \) and \( t = 15 \) in corollary 1 and corollary 4 respectively.

**Example 1** Let \( \zeta: \mathbb{R} \rightarrow \mathbb{R} \) be a function defined by

\[ \zeta(b) = \begin{cases} \gamma b^{16}, & \text{if } |b| < 1 \\ \gamma b^0, & \text{otherwise} \end{cases} \]

where \( \gamma_0 > 0 \) is a constant, and define a function \( h: \mathbb{R} \rightarrow \mathbb{R} \) by

\[ f(b) = \sum_{n=0}^{\infty} \frac{\zeta(2^n b)}{2^{16n}} \]

for all \( b \in \mathbb{R} \). Then \( f \) satisfies the inequality

\[ |G(f(b, c))| \leq \frac{(2092278995000000)65535}{(2092278995000000)65535}2\epsilon(|b|^{16} + |c|^{16}) \]

(40)

for all \( b, c \in \mathbb{R} \).

**Proof.** It is easy to see that \( h \) is bounded by \( \frac{65536}{65535} \) on \( \mathbb{R} \).

If \(|b|^{16} + |c|^{16} = 0\), then (40) is trivial.

If \(|b|^{16} + |c|^{16} \geq \frac{1}{2^{15}}\), then L.H.S of (40) is less than \( \frac{2092278995000000}{65535}\epsilon \).

Suppose that \( 0 < |b|^{16} + |c|^{16} < \frac{1}{2^{15}} \), then there exists a non-negative integer \( k \) such that

\[ \frac{1}{2^{16k+1}} \leq |b|^{16} + |c|^{16} < \frac{1}{2^{15}} \]

(41)

so that

\[ 2^{16(k+1)}|b|^{16} < \frac{1}{2^{16}} \cdot 2^{16(k+1)}|c|^{16} < \frac{1}{2^{15}} \]

and

\[ 2^n(c), 2^n(b \pm 6c), 2^n(b \pm 5c), 2^n(b \pm 4c), 2^n(b \pm 3c), 2^n(b \pm 7c), 2^n(b \pm 2c), 2^n(b), 2^n(b \pm 8c) \in (-1,1) \]

for all \( n = 0, 1, 2, \ldots, k - 1 \).

Hence \( G(h(2^n b, 2^n c)) = 0 \) for \( n = 0, 1, 2, \ldots, k - 1 \).

From the definition of \( h \) and (41), we get

\[ |Gf(b, c)| \leq \sum_{n=0}^{\infty} \frac{1}{2^{16n}}|Gf(2^n b, 2^n c)| \leq \frac{(2092278995000000)65535}{(2092278995000000)65535}2\epsilon(|b|^{16} + |c|^{16}) \]

**Example 2** Let \( \zeta: \mathbb{R} \rightarrow \mathbb{R} \) be a function defined by

\[ \zeta(b) = \begin{cases} \gamma b^{15}, & \text{if } |b| < 1 \\ \gamma b^0, & \text{otherwise} \end{cases} \]

where \( \gamma_0 > 0 \) is a constant, and define a function \( h: \mathbb{R} \rightarrow \mathbb{R} \) by

\[ f(b) = \sum_{n=0}^{\infty} \frac{\zeta(2^n b)}{2^{15n}} \]

for all \( b \in \mathbb{R} \). Then \( f \) satisfies the inequality

\[ |Df(b, c)| \leq \frac{(1307674400768)32767}{(1307674400768)32767}2\epsilon(|b|^{15} + |c|^{15}) \]

(40)

for all \( b, c \in \mathbb{R} \).

**Proof.** The proof is similar to the proof of Example 1.

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REFERENCES