A study on Zero divisor graph of the commutative Ring $Z_{2^k}$

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Abstract:

Inter relation between algebraic structures and graph theory is interesting and it facilitates to study the properties of graphs and vice versa. In this paper the zero divisor graph of the commutative ring on integer modulo $2^k$ is studied. The properties like planarity, Hamiltonian, Eulerian and parameters like radius, diameter, girth, domination number, degree of each vertex, total number of edges of the generated graphs are discussed in detail.

Key words: Zero divisor graph, degree, radius, diameter, girth, domination number

1. Introduction

In the field of Algebraic graph theory; the graph-theoretic problems are solved using the algebraic methods, mainly those employed in group theory for studying graph symmetry and linear algebra for spectral theory. Various types of graphs are defined based on the algebraic structures groups, rings, fields, integral domains and so on and their properties are studied. One such graph namely, zero divisor graph of a commutative ring was first introduced by Beck [5] in which the vertices of this graph are all elements in the ring $R$ and two vertices $x$, $y$ are adjacent if and only if $xy = 0$. This definition was revised by Anderson and Livingston [3] with the constraint that $x \neq 0, y \neq 0$ but $xy = 0$. Akbari et al shown that the edge chromatic number of a zero divisor graph $\Gamma(R)$ on any finite commutative ring $R$ is equal to maximum degree of $\Gamma(R)$, unless $\Gamma(R)$ is a complete graph of odd order. Alen Duric et al [2] characterized Artinian rings with the connected total zero-divisors graph and proved that the total zero-divisor graphs of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are isomorphic if and only if $m = n$. Cameron Wickham [8] proved that for a fixed positive integer $g$, there are finitely many isomorphism classes of rings whose zero-divisor graph has genus $g$. Darrin Weber[9], studied zero-divisor graph $\Gamma(R)$ and the compressed zero-divisor graph $\Gamma_c(R)$ of a finite commutative ring $R$ and a newly-defined graphical structure the zero-divisor lattice $\Lambda(R)$ of $R$. David et al [10] discussed about the ring theoretic properties and the graph properties of zero divisor graph. Khalida Nazzal and Manal Ghanem [11] studied about various properties of the zero divisor graphs on the set of Gaussian integers modulo $n$. The zero-divisor graph of a commutative ring has been studied extensively by many authors [1, 4, 6, 7].

In this paper, based on the zero divisor graph proposed by Anderson et al, the zero divisor graph of the commutative ring on integer modulo $2^k$ is defined and studied in detail. This paper is organized in such a manner that, section 1 deals with literature survey, 2 with the construction of the simple zero divisor graph, 3 with the properties and parameters of the graph, and finally the conclusion.
2. Construction of zero divisor graph on \(Z(Z_{2^k})\)

2.1 Definition: Zero divisor graph on commutative ring modulo \(2^k\)

Zero divisor graph of a commutative ring on \(Z(Z_{2^k})\) is a graph whose vertex set is the set of all zero divisors \(Z(Z_{2^k}) = \{2, 4, \ldots, 2^k - 2\}\) and two distinct vertices \(x \neq 0, y \neq 0\) are adjacent if and only if \(xy \equiv 0 \pmod{2^k}\).

Sketching of zero divisor graph on \(Z(Z_{2^k})\)

The elements of commutative ring of integer modulo \(2^k\) is \(Z_{2^k} = \{0, 1, 2, \ldots, 2^k - 1\}\) and its zero divisors are given by \(Z(Z_{2^k}) = \{2, 4, \ldots, 2^k - 2\}\). Also \(|Z(Z_{2^k})| = 2^{k-1} - 1\).

- With zero divisors as vertices the zero divisor graph is drawn as follows. The vertices set \(\{2, 4, \ldots, 2^k - 2\}\) is partitioned into \((k-1)\) sets \(V_1 = \{2n_1 / n_1 \text{ is an odd number } < 2^k-1\}\); \(V_2 = \{2^2n_2 / n_2 \text{ is an odd number } < 2^{k-2}\}\); \(V_3 = \{2^3n_3 / n_3 \text{ is an odd number } < 2^{k-3}\}\); \ldots; \(V_{k-1} = \{2^{k-1}\}\).
- To construct the zero divisor graph, the vertex set \(V_1\) is kept at first level, \(V_{k-1}\) is kept at second level, \(V_2\) at third level, \(V_3\) at fourth level, \ldots, and \(V_{k-2}\) at \(k-1\)th level.
- Any two vertices \(a, b \in Z(Z_{2^k})\) are connected by an edge if and only if \(ab \equiv 0 \pmod{2^k}\).

**Example 1:** The zero divisors of \(Z_{16}\) are given by \(Z(Z_{16}) = \{2, 4, 6, 8, 10, 12, 14\}\). The zero divisor graph can be constructed with the partition \(V_1 = \{2, 6, 10, 14\}\) at first level, \(V_2 = \{8\}\) at the second level, \(V_3 = \{4, 12\}\) at third level according to the relation \(ab \equiv 0 \pmod{16}\), for \(a, b \in Z(Z_{16})\) as follows.

![Zero divisor graph of \(Z_{16}\)](image)

**Example 2:** The zero divisors \(Z_{32}\) are given by \(\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 28, 32\}\) and it can be partitioned into \(V_1 = \{2, 6, 10, 14, 18, 22, 26, 30\}\); \(V_2 = \{4, 12, 20, 28\}\); \(V_3 = \{8, 24\}\); \(V_4 = \{16\}\) and the zero divisor graph is given by
Example 3: The zero divisors of $\mathbb{Z}_{64}$ are given by $\{2, 4, 6, \ldots, 62\}$ can be partitioned into $V_1 = \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62\}$; $V_2 = \{4, 12, 20, 28, 36, 44, 52, 60\}$; $V_3 = \{8, 24, 40, 56\}$; $V_4 = \{16, 48\}$; $V_5 = \{32\}$ and the zero divisor graph is given by

Observations:
- Cardinality of the partitions of $Z(2^k)$ are $|V_1| = 2^{k-2}$, $|V_2| = 2^{k-3}$, \ldots, $|V_{k-2}| = 2$, $|V_{k-1}| = 1$.
- Any level, with the labeling of nodes that are multiples of $2^x$ consists of $2^{k-1-x}$ nodes.
- The zero divisor graph on $Z_{2^k}$ is simple.
• On visualizing the structure, this kind of graphs could be named Flower Vase graph.

### 3. Properties of zero divisor graphs of $Z_{2^k}$

In this section some of the properties and parameters associated with the zero divisor graphs are discussed.

**Theorem 1:** The number of pendant vertices of the zero divisor graph $Z_{2^k}$ is $2^{k-2}$

**Proof:** For the partition $V_{4} = \{2n_1 / n_1 \text{ is an odd number } < 2^{k-1}\}$, for every $a \in V_1$ there exists only one partition $V_{k-1} = \{2^{k-1}\}$ such that for $b \in V_{k-1}$, and $ab \equiv 0(mod 2^k)$. For all elements $c$ in the remaining partitions $ac \neq 0(mod 2^k)$ is satisfied. Thus there will be exactly $2^{k-2}$ pendant vertices with labeling $[2n_1 / n_1 \text{ is an odd number } < 2^{k-1}]$.

**Example:** The zero divisor graphs in Example 1 and 2 have 4, 16 pendent vertices with labeling namely $\{2, 6, 10, 14\}$ and $\{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62\}$ respectively.

**Theorem 2:** Degree of the vertex in the second level of the zero divisor graph of $Z_{2^k}$ is $2^{k-1} - 1$

**Proof:** The second level vertex is the partition set $V_{k-1} = \{2^{k-1}\}$, and it is adjacent to all vertices of the zero divisor graph. Since for $a \in V_{k-1}$ and for every vertex $b$ in the partitions $V_1, V_2, ..., V_{k-2}$, $ab \equiv 0(mod 2^k)$. Thus degree of $V_{k-1}$ is $|Z(Z_{2^k}) - 1| = 2^{k-1} - 2$.

**Theorem 3:** For the zero divisor graph on $Z_{2^k}$, in the $r^{th}$ level for $r \geq 2$

(i) degree of each vertex in $V_{r-1} = \sum_{i=1}^{r-1} |V_{k-1}|$ if $ab \neq 0(mod 2^k)$ for $a, b \in V_{r-1}$.

(ii) degree of each vertex in $V_{r-1} = \sum_{i=1}^{r-1} |V_{k-1}| - 1$ if $ab \equiv 0(mod 2^k)$ for $a, b \in V_{r-1}$.

**Proof:** (i) If $ab \neq 0(mod 2^k)$ for $a, b \in V_{r-1}$, then the vertices of $V_{r-1}$ are adjacent to the vertices of the distinct partitions $V_{k-1}, V_{k-2}, ..., V_{k-r+1}$. The labeling of vertices of $V_{r-1}$ consists of $2^{r-1}$ as a factor and the labeling of vertices of $V_{k-1}, V_{k-2}, ..., V_{k-r+1}$ consists of $2^{k-1}, 2^{k-2}, ..., 2^{k-r+1}$ as factors respectively. Therefore $ac \equiv 0(mod 2^k)$ for every $a \in V_{r-1}$ and for every $c$ in $V_{k-1}, V_{k-2}, ..., V_{k-r+1}$. Thus degree of each vertex in $V_{r-1}$ is $|V_{k-1}| + |V_{k-2}| + ... + |V_{k-r+1}|$. Hence degree of each vertex in $V_{r-1} = \sum_{i=1}^{r-1} |V_{k-1}|$

(ii) If $ab \equiv 0(mod 2^k)$ for $a, b \in V_{r-1}$, in addition to case (i), distinct vertices of $V_{r-1}$ are adjacent to distinct vertices of $V_{r-1}$ except to themselves. Thus we get $V_{r-1} = \sum_{i=1}^{r-1} |V_{k-1}| - 1$ if $ab \equiv 0(mod 2^k)$ for $a, b \in V_{r-1}$.

**Examples:**

For divisor graph of $Z_{16}$, each $3^{rd}$ level vertex is of degree $= \sum_{i=1}^{2} |V_{k-1}| = 1 = |V_3| + |V_2| - 1 = 1 + 2 - 1 = 2$

For divisor graph of $Z_{32}$, each $3^{rd}$ level vertex is of degree $= \sum_{i=1}^{2} |V_{k-1}| = |V_4| + |V_3| = 1 + 2 = 3$
For divisor graph of $\mathbb{Z}_{32}$, each 4th level vertex is of degree $\sum_{i=1}^{3} |V_{k-i}| - 1 = |V_4| + |V_3| + |V_2| - 1 = 1 + 2 + 4 - 1 = 6$

For divisor graph of $\mathbb{Z}_{64}$, each 3rd level vertex is of degree $\sum_{i=1}^{2} |V_{k-i}| = |V_5| + |V_4| = 1 + 2 = 3$

For divisor graph of $\mathbb{Z}_{64}$, each 4th level vertex is of degree $\sum_{i=1}^{3} |V_{k-i}| - 1 = |V_5| + |V_4| + |V_3| - 1 = 1 + 2 + 4 - 1 = 6$

For divisor graph of $\mathbb{Z}_{64}$, each 5th level vertex is of degree $\sum_{i=1}^{3} |V_{k-i}| - 1 = |V_5| + |V_4| + |V_3| + |V_2| - 1 = 1 + 2 + 4 + 8 - 1 = 14$

**Theorem 4:** For the zero divisor graph on $\mathbb{Z}_{2^k}$

(i) the radius is 1

(ii) the diameter is 2

(iii) the girth is 3.

(iv) centre of the graph is $V_{k-1} = \{2^{k-1}\}$

**Proof:**

(i) The vertex of the partition $V_{k-1} = \{2^{k-1}\}$ is adjacent to every vertex of the graph and thus it has the eccentricity 1. Hence the radius of the graph is always 1.

(ii) Every vertex can be reachable to every other vertex through the partition $V_{k-1} = \{2^{k-1}\}$ of distance 2. Hence diameter of the graph is 2.

(iii) In the graph, always the last level of vertices is adjacent among themselves and also the previous level of vertices. Thus always a triangle will be formed at the last two levels. Hence the girth is 3.

(iv) The only vertex with minimum eccentricity equal to the radius 1 is $V_{k-1} = \{2^{k-1}\}$, since it is adjacent to every vertex of the graph. Thus, the centre of the graph is $V_{k-1} = \{2^{k-1}\}$.

**Theorem 5:** The total number of edges in a zero divisor graph on $\mathbb{Z}_{2^k}$ is

$$2(2^{k-2} - 1) + 2^3(2^{k-4} - 1) + 2^5(2^{k-6} - 1) + \cdots + 1! + 3! + \cdots + \left(2^{k-\left\lceil \frac{k}{2} \right\rceil} - 1\right)!$$

**Proof:**

**Case(i):** $k$ is odd

In the simple zero divisor graph on $\mathbb{Z}_{2^k}$, there will be adjacency between the levels as follows

<table>
<thead>
<tr>
<th>With the labeling of From nodes</th>
<th>With the labeling of To nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodes with labeling multiple of $2^1$</td>
<td>Nodes with labeling multiple of $2^{k-1}$</td>
</tr>
<tr>
<td>Nodes with labeling multiple of $2^{k-1}$</td>
<td>Nodes with labeling multiple of $2^2$, Nodes with labeling multiple of $2^3$, $\cdots$, Nodes with labeling multiple of $2^{k-2}$</td>
</tr>
<tr>
<td>Nodes with labeling multiple of $2^{k-2}$</td>
<td>Nodes with labeling multiple of $2^2$,</td>
</tr>
</tbody>
</table>
For any two nodes with labeling ‘a’ from the first column and with labeling ‘b’ from the second column, \( ab \equiv 0 \mod 2^k \).

In the above table the number of nodes in **From** level and the number of nodes in **To** level are as follows

<table>
<thead>
<tr>
<th>the number of nodes in From level</th>
<th>the number of nodes in To level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of nodes with labeling multiple of ( 2^1 = 2^k )</td>
<td>Number of nodes with labeling multiple of ( 2^{k-1} = 1 )</td>
</tr>
<tr>
<td>Number of nodes with labeling multiple of ( 2^{k-1} = 1 )</td>
<td>Number of nodes with labeling multiple of ( 2^k = 2^k ), Number of nodes with labeling multiple of ( 2^3 = 2^{k-4} ), …, Number of nodes with labeling multiple of ( 2^{k-2} = 2 )</td>
</tr>
<tr>
<td>Number of nodes with labeling multiple of ( 2^{k-2} = 2 )</td>
<td>Number of nodes with labeling multiple of ( 2^3 = 2^{k-4} ), Number of nodes with labeling multiple of ( 2^3 = 2^{k-4} ), \ldots, Number of nodes with labeling multiple of ( 2^{k-3} = 2^2 )</td>
</tr>
<tr>
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</tr>
<tr>
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<td>Number of nodes with labeling multiple of ( 2^{k-5} = 2^{k-5} )</td>
</tr>
</tbody>
</table>

Thus, the number of between edges in the above levels

\[
2^{k-2} + 2^{k-3} + \ldots + 2 + 2\left(2^2 + 2^3 + \ldots + 2^{k-3}\right) + 2^2\left(2^3 + 2^4 + \ldots + 2^{k-4}\right) + \ldots + 2^{k-4}\left(2^{k-4} - 1\right) + 2^{k-5}\left(2^{k-6} - 1\right) + \ldots
= 2(2^{k-2} - 1) + 2^3(2^{k-4} - 1) + 2^5(2^{k-6} - 1) + \ldots
\]

Also, In the zero divisor graph on \( Z_{2^k} \), there will be adjacency at the same levels with the nodes having labeling multiples of \( 2^{[k/2]} \), \( 2^{[k/2]}+1 \), \ldots, \( 2^{k-2} \), since for any nodes with labeling ‘a’ in these levels \( a^2 \equiv 0 \mod 2^k \). The number of nodes in each of these levels are \( 2^{k-\left[\frac{k}{2}\right]-1} \), \( 2^{k-\left[\frac{k}{2}\right]-2} \), \ldots, \( 2^2 \), \( 2 \) respectively. Thus, the number of edges at same levels is \( \left(2^{k-\left[\frac{k}{2}\right]-1} - 1\right)! + \left(2^{k-\left[\frac{k}{2}\right]-2} - 1\right)! + \ldots + 3! + 1! \)

Hence the total number of edges in the zero divisor graph on \( Z_{2^k} \) is

\[
2(2^{k-2} - 1) + 2^3(2^{k-4} - 1) + 2^5(2^{k-6} - 1) + \ldots + 1! + 3! + \ldots + \left(2^{k-\left[\frac{k}{2}\right]-1} - 1\right)!
\]

**Case (ii):** \( k \) is even

In the simple zero divisor graph on \( Z_{2^k} \), there will be adjacency between the levels as follows
<table>
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</tr>
<tr>
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<td>Nodes with labeling multiple of $2^4$, ..., Nodes with labeling multiple of $2^{k-5}$</td>
</tr>
<tr>
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</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Nodes with labeling multiple of $2^{</td>
<td>k/2</td>
</tr>
</tbody>
</table>

For any two nodes with labeling ‘a’ from the first column level and with labeling ‘b’ from the second column level, $ab \equiv 0 (mod 2^k)$.

In the above table the number of nodes in From level and the number of nodes in To level are as follows

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</tr>
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Thus, the number of between edges in the above levels

\[
2^{k-2} + 2^{k-3} + \cdots + 2 + 2^2 (2^2 + 2^3 + \cdots + 2^3) + 2^3 (2^3 + 2^4 + \cdots + 2^4) + \cdots + 2^{k-\lfloor k/2 \rfloor - 2} 2^{k-\lfloor k/2 \rfloor} \\
= 2(2^{k-2} - 1) + 2^3(2^{k-4} - 1) + 2^5(2^{k-6} - 1) + \cdots
\]

As in the previous case, the number of edges at same levels is

\[
\left(2^{k-\lfloor k/2 \rfloor-1} - 1\right)! + \left(2^{k-\lfloor k/2 \rfloor-2} - 1\right)! + \cdots + 3! + 1
\]
Hence, the total number of edges in the simple zero divisor graph on $Z_{2^k}$ is
\[ 2(2^{k-2} - 1) + 2^2(2^{k-4} - 1) + 2^5(2^{k-6} - 1) + \ldots + 1! + 3! + \ldots + \left(2^{k-\left\lfloor \frac{k}{2} \right\rfloor} - 1\right)! \]

**Example:**

For $Z_{16}$, $k = 4$ the total number of edges = $2(3) + 1! = 7$

For $Z_{32}$, $k = 5$ the total number of edges = $2(7) + 8(1) + 1! = 23$

For $Z_{64}$, $k = 6$ the total number of edges = $2(15) + 8(3) + 1! + 3! = 61$

**Theorem 6:** The degree sequence of the zero divisor graph on $Z_{2^k}$ for $k > 4$ is

(i) $\{1, 1, \ldots 2^{k-2} \text{ times}, 3, 3, \ldots 2^{k-3} \text{ times}, 6, 6, \ldots 2^{k-4} \text{ times}, \ldots, (2^{k-\left\lfloor \frac{k}{2} \right\rfloor} - 1), (2^{k-\left\lfloor \frac{k}{2} \right\rfloor} - 1), \ldots 2^{k-\left\lfloor \frac{k}{2} \right\rfloor} \text{ times}, (2^{k-\left\lfloor \frac{k}{2} \right\rfloor} - 2), (2^{k-\left\lfloor \frac{k}{2} \right\rfloor} - 2), \ldots 2^{k-\left\lfloor \frac{k}{2} \right\rfloor} \text{ times}, \ldots, 2(2^{k-3} - 1), 2(2^{k-3} - 1), 2(2^{k-2} - 1)\};$ when $k$ is odd.

(ii) $\{1, 1, \ldots 2^{k-2} \text{ times}, 3, 3, \ldots 2^{k-3} \text{ times}, 6, 6, \ldots 2^{k-4} \text{ times}, \ldots, (2^{k-\left\lfloor \frac{k}{2} \right\rfloor} - 2), (2^{k-\left\lfloor \frac{k}{2} \right\rfloor} - 2), \ldots \}
\[2^{k-1-\left\lfloor k/2 \right\rfloor} \text{ times}, \ldots, 2(2^{k-3} - 1), 2(2^{k-3} - 1), 2(2^{k-2} - 1)\};$ when $k$ is even.

**Proof:**

**Case(i):** $k$ is odd

- In the zero divisor graph on $Z_{2^k}$, $2^{k-2}$ pendant vertices with labeling multiple of 2 are adjacent to vertex with labeling $2^{k-1}$ and thus pendant vertices have degree sequence $\{1, 1, \ldots, 2^{k-2} \text{ times}\}$.
- The vertex with labeling $2^{k-1}$ is adjacent to all vertices of the graph and thus it has the degree $2 + 2^2 + \ldots + 2^{k-2} = 2(2^{k-2} - 1)$.
- $2^{k-3}$ vertices with labeling multiples of 2 are adjacent to a vertex with labeling $2^{k-1}$ and to 2 vertices with labeling multiples of $2^{k-2}$ and thus these $2^{k-3}$ vertices have degree sequence $\{3, 3, \ldots 2^{k-3} \text{ times}\}$.
- Proceeding, $2^{k-1-\left\lfloor k/2 \right\rfloor}$ vertices with labeling $2^{\left\lfloor k/2 \right\rfloor}$ are adjacent to a vertex with labeling $2^{k-1}$, to 2 vertices with labeling multiples of $2^{k-2}$, to $2^2$ vertices with labeling $2^{k-3}$, … and to $2^{k-1-\left\lfloor k/2 \right\rfloor}$ vertices with labeling $2^{\left\lfloor k/2 \right\rfloor}$. Thus, these $2^{k-1-\left\lfloor k/2 \right\rfloor}$ vertices have the degree as $1 + 2 + 2^2 + \ldots + 2^{k-1-\left\lfloor k/2 \right\rfloor} = 2^{k-\left\lfloor k/2 \right\rfloor} - 1$ and the degree sequence is $\{2^{k-\left\lfloor k/2 \right\rfloor} - 1, 2^{k-\left\lfloor k/2 \right\rfloor} - 1, \ldots, 2^{k-1-\left\lfloor k/2 \right\rfloor} \text{ times}\}$
- $2^{k-1-\left\lfloor k/2 \right\rfloor}$ vertices with labeling multiples of $2^{\left\lfloor k/2 \right\rfloor}$ are adjacent to a vertex with labeling $2^{k-1}$, to 2 vertices with labeling multiples of $2^{k-2}$, to $2^2$ vertices with labeling $2^{k-3}$, …, to $2^{k-1-\left\lfloor k/2 \right\rfloor} - 1$ vertices with labeling multiples of $2^{\left\lfloor k/2 \right\rfloor}$ at same level and to $2^{k-1-\left\lfloor k/2 \right\rfloor}$ vertices with labeling $2^{\left\lfloor k/2 \right\rfloor}$. Thus these $2^{k-1-\left\lfloor k/2 \right\rfloor}$ vertices have the degree as $1 + 2 + 2^2 + \ldots + 2^{k-1-\left\lfloor k/2 \right\rfloor} - 1 + 2^{k-1-\left\lfloor k/2 \right\rfloor} + = 2^{k-\left\lfloor k/2 \right\rfloor} - 2$ and the degree sequence is $\{2^{k-\left\lfloor k/2 \right\rfloor} - 2, 2^{k-\left\lfloor k/2 \right\rfloor} - 2, \ldots, 2^{k-1-\left\lfloor k/2 \right\rfloor} \text{ times}\}$
- Finally, 2 vertices with labeling multiples of $2^{k-2}$ are adjacent to each other at same level and also to all other vertices of the graph except $2^{k-2}$ pendant vertices. Thus these 2 vertices have the
degree as \( 1 + 1 + 2^2 + 2^3 + \cdots + 2^{k-3} = 2(2^{k-3} - 1) \), and thus the degree sequence is \( \{2(2^{k-3} - 1), 2(2^{k-3} - 1)\} \)

Hence, if \( k \) is odd, the degree sequence of the zero divisor graph on \( Z_{2k} \) is \( \{1, 1, \ldots, 2^{k-2} \text{ times}, 3, 3, \ldots, 2^{k-3} \text{ times}, 6, 6, \ldots, 2^{k-4} \text{ times} \ldots, (2^{k-[k/2]} - 1), (2^{k-[k/2]} - 1), \ldots, (2^{k-[k/2]} - 1) \text{ times}, (2^{k-[k/2]} - 2), (2^{k-[k/2]} - 2), \ldots, 2^{k-1-[k/2]} \text{ times}, \ldots, 2(2^{k-3} - 1), 2(2^{k-3} - 1), 2(2^{k-2} - 1)\} \)

**Case (ii): \( k \) is even**

- In the zero divisor graph on \( Z_{2k} \), \( 2^{k-2} \) pendant vertices with labeling multiple of 2 are adjacent to vertex with labeling \( 2^{k-1} \) and thus pendant vertices have degree sequence \( \{1, 1, \ldots, 2^{k-2} \text{ times}\} \).
- The vertex with labeling \( 2^{k-1} \) is adjacent to all vertices of the graph and thus it has the degree \( 2 + 2^2 + \cdots + 2^{k-2} = 2(2^{k-2} - 1) \).
- \( 2^{k-3} \) vertices with labeling multiples of \( 2^2 \) are adjacent to a vertex with labeling \( 2^{k-1} \) and to \( 2 \) vertices with labeling multiples of \( 2^{k-2} \) and thus these \( 2^{k-3} \) vertices have degree sequence as \( \{3, 3, \ldots, 2^{k-3} \text{ times}\} \).
- Proceeding, \( 2^{k-1-[k/2]} \) vertices with labeling \( 2^{k-[k/2]} \) are adjacent to a vertex with labeling \( 2^{k-1} \), to \( 2 \) vertices with labeling multiples of \( 2^{k-2} \), to \( 2^2 \) vertices with labeling \( 2^{k-3} \), \ldots and to \( 2^{k-2-[k/2]} \) vertices with labeling \( 2^{k-[k/2] + 1} \), to \( 2^{k-1-[k/2]} - 1 \) vertices at same level. Thus these vertices have degree as \( 1 + 2 + 2^2 + \cdots + 2^{k-2-[k/2]} + 2^{k-1-[k/2]} - 1 = 2^{k-[k/2]} - 1 \).
- Finally, \( 2 \) vertices with labeling multiples of \( 2^{k-2} \) are adjacent to each other at same level and also to all other vertices of the graph except \( 2^{k-2} \) pendant vertices. Thus these \( 2 \) vertices have the degree as \( 1 + 1 + 2^2 + 2^3 + \cdots + 2^{k-3} = 2(2^{k-3} - 1) \), and thus the degree sequence is \( \{2(2^{k-3} - 1), 2(2^{k-3} - 1)\} \)

Hence when \( k \) is even, the degree sequence of the zero divisor graph on \( Z_{2k} \) is \( \{1, 1, \ldots, 2^{k-2} \text{ times}, 3, 3, \ldots, 2^{k-3} \text{ times}, 6, 6, \ldots, 2^{k-4} \text{ times} \ldots, (2^{k-[k/2]} - 2), (2^{k-[k/2]} - 2), \ldots, 2^{k-1-[k/2]} \text{ times}, \ldots, 2(2^{k-3} - 1), 2(2^{k-3} - 1), 2(2^{k-2} - 1)\} \); when \( k \) is even.

**Example:**

For the zero divisor graph on \( Z_{2^5} \), the degree sequence is \( \{1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 6, 6, 14\} \)

For the zero divisor graph on \( Z_{2^6} \), the degree sequence is \( \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 6, 6, 6, 6, 14, 14, 14, 30\} \)

**Theorem 7:** The zero divisor graph on \( Z_{2k} \), for \( k>5 \) is non-planar.

**Proof:** For the zero divisor graph on \( Z_{2k} \), with \( k>5 \), in \( k \)-th level there are 4 vertices with labeling which are multiples of \( 2^{k-3} \) and in \( k \)-th level there are 2 vertices with labeling which are multiples of \( 2^{k-2} \). Thus, the product \( 2^{k-3}, 2^{k-2} \equiv 0 \text{(mod } 2k) \). Thus these 6 vertices are adjacent to each other. It is possible to extract a complete graph \( K_5 \) from the last two levels. Hence the simple zero divisor graph on \( Z_{2k} \), with \( k>5 \) is not a planar graph.
**Theorem 8:** The domination number of the zero divisor graph is $\gamma(G) = 1$.

**Proof:** Since vertex corresponding to $\{2^{k-1}\}$ is adjacent to every vertex of the zero divisor graph, the minimum dominating set of the zero divisor graph of $Z_{2^k}$ is $V_{k-1}$ and hence the domination number is $\gamma(G) = 1$.

**Observations:**
- The simple zero divisor graph on $Z_{2^k}$ is neither Eulerian nor Hamiltonian.
- The simple zero divisor graph on $Z_{2^k}$, for $k<5$ are planar graph.

**Conclusion:** In this paper, we defined the zero divisor graph on the commutative ring of integer modulo $Z_{2^k}$ and discussed their structural properties. Also some interesting observations about the graph are stated. Results on other properties of such graphs are also given which are supported by theoretical proofs. Further, our future enhancement is to find out other parameters like cliques, chromatic number of the zero divisor graph on $Z_{2^k}$.

**References:**


