

Application of ADM For Solving Fractional Heat like Equations Using Natural Transform

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Abstract

In this paper, we apply an algorithm to find the exact and approximate solution of nonlinear fractional heat like equations. The algorithm is the combination of two powerful methods namely Natural transform and Adomian Decomposition Method which can be applied to fractional partial differential equations to obtain the approximate solution. The efficiency and accuracy of algorithm is illustrated by solving some example.

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1 Introduction

The exhaustive literature survey reveals that, the fractional calculus attracts many researchers in this field due to the wide range of applications in the field of applied science and engineering. The fields like visco-elasticity, biology, signal processing, optics, fluid mechanics etc. have their applications in fractional calculus. The branch of physical sciences includes the linear and non-linear fractional differential equations. In the recent years many works have been done on the fractional calculus with wide range of applications (I. Podlubny, 1999; K.B. Oldham, J. Spanier, 1974; K.S. Miller, B. Ross, 1993; Schneider, W., Wyss, W., 1989).

The main objective of study of fractional calculus is to obtain exact and approximate solution of the linear and non-linear fractional differential equations. The various analytical and numerical methods have been developed to get exact

and approximate solution of linear and non linear fractional differential equations. Some of them are Variational Iteration Method, Adomian Decomposition Method, Differential Transform Method, Homotopy Perturbation Method etc. (S. Momani, Z. Odibat, 2007; Q. Wang, 2007; Q. Wang, 2008; J.H. He, 1999; J.K. Zhou, 1986; A. Al-rabtah, V.S. Ertürk. Momani, S., 2010; Y. Khan, Q. B. Wu., 2011; ?; V. Daftardar-Gieji, H. Jafari, 2007; E. Yusufoglu, 2006). The aim of this paper is to modify the ADM by combining it with the new integral transform Natural transform to obtain the approximate solution of fractional heat-like equations. This method is illustrated with some numerical example. The heat-like equations can be seen in the field of science and engineering. The presented fractional heat like equations has been applied in modeling to describe practical sub diffusion problems in fluid flow process and finance.

1.1 Fractional Calculus

In this section some basic definitions and properties of fractional calculus are presented.

Definition 1.1. A real function $f(x), x > 0$ is said to be in $\mathbb{C}_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in \mathbb{C}[0, \infty]$, and it is said to be in the space \mathbb{C}_μ^m if and only if $f^{(m)} \in \mathbb{C}_\mu, m \in \mathbb{N}$

Definition 1.2. The Riemann Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in \mathbb{C}_\mu$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, x > 0$$

$$J^0 f(x) = f(x)$$

The properties of operator J^α can be seen in (G. Jumarie, 200), some of them are

$$(1) J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$$

$$(2) J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$

$$(3) J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

For $f \in \mathbb{C}_\mu, \mu \geq -1, \alpha, \beta \geq 0, \gamma > -1$

Definition 1.3. The fractional derivative of $f(x)$ in Caputo sense is defined by (M. Caputo, 1997)

$$D_*^\alpha f(x) = J^{m-\alpha} D_*^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt$$

for $m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0, f \in \mathbb{C}_{-1}^m$

Definition 1.4. Suppose that $f(x)$ is the function of n variables $x_i, i = 1, 2, \dots, n$ and of class C on $D \in \mathbb{R}_n$ then

$$a \partial_x^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_0^{x_i} (x_i-t)^{m-\alpha-1} \partial_{x_i}^\alpha f(x_j) \Big|_{x_j=t} dt$$

where ∂_x^α is the usual partial derivative of integer order m .

Definition 1.5. The Natural transform of Caputo fractional derivative is defined by equation

$$\mathbb{N}[D_t^\alpha f(t)] = \frac{s^\alpha}{u^\alpha} R(s, u) - \sum_{k=0}^n \frac{s^{n-k-1}}{u^{n-k}} f^{(k)}(0) \quad n-1 \leq \alpha < 1$$

Lemma 1.1. For $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, $f \in \mathbb{C}_{-1}^m$ and $\mu \geq -1$, then

$$D_*^\alpha J^\alpha f(x) = f(x)$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!} \quad x > 0$$

1.2 Natural Transform

The new integral transform Natural transform was established by Khan and Khan (Khan Z.H., Khan W.A., 2008) as N-transform who studied its properties and application. The Natural transform of the function $f(t) \in \mathbb{R}^2$ is denoted by symbol $\mathbb{N}[f(t)] = R(s, u)$ where s and u are the transform variables and is defined by an integral equation

$$\mathbb{N}[f(t)] = R(s, u) = \int_0^\infty e^{-st} f(ut) dt \quad (1.1)$$

where $\operatorname{Re}(s) > 0$, $u \in (\tau_1, \tau_2)$, the function $f(t)$ is sectionwise continuous, exponential order and defined over the set $A = [f(t)/\exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty)]$.

The above equation can be written in another form as

$$\mathbb{N}[f(t)] = R(s, u) = \frac{1}{u} \int_0^\infty e^{-\frac{st}{u}} f(t) dt \quad (1.2)$$

The inverse Natural transform of function $R(s, u)$ is denoted by symbol $\mathbb{N}^{-1}[R(s, u)] = f(t)$ and is defined with Bromwich contour integral (Belgacem FBM, 2011; Belgacem FBM, 2012)

$$\mathbb{N}^{-1}[R(s, u)] = f(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{\frac{st}{u}} R(s, u) ds \quad (1.3)$$

If $R(s, u)$ is the Natural transform, $F(s)$ is the Laplace transform and $G(u)$ is Sumudu transform of function $f(t) \in A$, then we can have Natural-Laplace and Natural-Sumudu duality as

$$\mathbb{N}[f(t)] = R(s, u) = \int_0^\infty e^{-st} f(ut) dt = \frac{1}{u} F\left(\frac{s}{u}\right) \quad (1.4)$$

and

$$\mathbb{N}[f(t)] = R(s, u) = \int_0^{\infty} e^{-st} f(ut) dt = \frac{1}{s} G\left(\frac{u}{s}\right) \quad (1.5)$$

We can extract the Laplace, Sumudu, Fourier and Mellin transform from Natural transform and which shows that Natural transform convergence to Laplace and Sumudu transform. Moreover Natural transform plays as a source for other transform and it is the theoretical dual of Laplace transform. Further study and applications of Natural transform can be seen in (Khan Z.H., Khan W.A., 2008; Deshna Looner and P.K. Banerji, 2013; Deshna Looner and P.K. Banerji, 2013; Chindhe A.D., Kiwne S.B., 2016; Chindhe A.D., Kiwne S.B., 2017).

1.3 Natural Transform of Some Standard Functions

$$1. \mathbb{N}[1] = \frac{1}{s}$$

$$2. \mathbb{N}[t^n] = \frac{u^n}{s^{n+1}} n!$$

$$3. \mathbb{N}[e^{at}] = \frac{1}{s - au}$$

$$4. \mathbb{N}\left[\frac{t^{n-1} e^{at}}{(n-1)!}\right] = \frac{u^{n-1}}{(s - au)^2}$$

$$5. \mathbb{N}[f^{(n)}(t)] = \frac{s^n}{u^n} R(s, u) - \sum_{k=0}^{\infty} \frac{s^{n-(k+1)}}{u^{n-k}} \cdot u^{(k)}(0)$$

6. If $F(s, u)$ and $G(s, u)$ are the Natural transforms of respective functions $f(t)$ and $g(t)$ both defined in set A then (Convolution)

$$\mathbb{N}[f * g] = u \cdot F(s, u) G(s, u)$$

2 Main Result

In this section we apply the Natural decomposition method to the fractional heat-like equations to obtain the approximate solution.

Let us consider the general non-linear non-homogeneous fractional partial differential equation of the form

$$D_t^\alpha [U(x, t)] = L[U(x, t)] + N[U(x, t)] + f(x, t), \quad \alpha > 0 \quad (2.1)$$

subject to the initial condition

$$D_0^k [U(x, 0)] = g_k, \quad k = 0, 1, 2, \dots, n-1$$

$$D_0^n [U(x, 0)] = 0, \quad n = [\alpha]$$

where D_t^α denotes without loss of generality the Caputo fractional derivative operator, f is known function, N is general nonlinear fractional differential operator and L is linear fractional differential operator.

Applying the Natural transform on both sides of equation (2.1)

$$\mathbb{N}[D_t^\alpha[U(x, t)]] = \mathbb{N}[L[U(x, t)]] + \mathbb{N}[N[U(x, t)]] + \mathbb{N}[f(x, t)]$$

$$\mathbb{N}[U(x, t)] = \frac{u^\alpha}{s^\alpha} \{ \mathbb{N}[L[U(x, t)]] + \mathbb{N}[N[U(x, t)]] + \mathbb{N}[f(x, t)] \} + \frac{u^\alpha}{s^\alpha} \sum_{k=0}^n \frac{s^{n-k-1}}{u^{n-k}} U^{(k)}(0)$$

Now operating the inverse Natural transform on both sides we obtain,

$$U(x, t) = F(x, t) + \mathbb{N}^{-1} \left\{ \frac{u^\alpha}{s^\alpha} \{ \mathbb{N}[L[U(x, t)]] + \mathbb{N}[N[U(x, t)]] \} \right\} \quad (2.2)$$

Where $F(x, t)$ is the term arising from the known function and the initial conditions. To find the approximate solution $U(x, t)$, we apply Adomian decomposition method for which consider

$$U(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad (2.3)$$

where the components $u_n(x, t)$ can be calculated by using recursive relation. However the non linear terms can be calculated by the adomian polynomials A_n in the form

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left(F \left(\sum_{n=0}^{\infty} \lambda^n U_n(x) \right) \right) \quad n = 0, 1, 2, \dots \quad (2.4)$$

∴ Equation (2.2) becomes

$$\sum_{n=0}^{\infty} u_n(x, t) = F(x, t) + \mathbb{N}^{-1} \left\{ \frac{u^\alpha}{s^\alpha} \{ \mathbb{N}[L[\sum_{n=0}^{\infty} u_n(x, t)]] + \mathbb{N}[N[\sum_{n=0}^{\infty} A_n]] \} \right\} \quad (2.5)$$

Here is the step where we are combining the two powerful methods, Natural transform and Adomian decomposition method. From this equation we get the recursive relation of the form

$$u_0(x, t) = F(x, t) \quad (2.6)$$

$$u_{k+1}(x, t) = \mathbb{N}^{-1} \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N} \left[\sum_{n=0}^{\infty} u_n(x, t) + A_k \right] \right\} \quad (2.7)$$

This gives us the series solution of the given general non-linear non-homogeneous fractional partial differential equation with the given initial condition.

3 Illustrative Examples

In this section, we solve some non-linear non-homogeneous fractional partial differential equation by using the Natural decomposition method.

Example: Consider the following two-dimensional fractional heat-like equation

$$D_t^\alpha u(x, y, t) = \frac{1}{2}(y^2 u_{xx} + x^2 u_{yy}) \quad 0 < \alpha \leq 1 \quad (3.1)$$

subject to initial condition

$$u(x, y, 0) = y^2 \quad (3.2)$$

subject to boundary condition

$$u_x(0, y, t) = 0, u_y(x, 0, t) = 0, u_x(1, y, t) = 2\text{Sinh}(t), u_y(x, 1, t) = 2\text{Cosh}(t) \quad (3.3)$$

Solution: Applying the Natural transform on both sides of given equation and using the initial condition

$$\begin{aligned} \mathbb{N}[D_t^\alpha u(x, y, t)] &= \mathbb{N}\left[\frac{1}{2}(y^2 u_{xx} + x^2 u_{yy})\right] \\ \mathbb{N}[u(x, y, t)] &= \frac{1}{s} y^2 + \frac{u^\alpha}{s^\alpha} \mathbb{N}\left[\frac{1}{2}(y^2 u_{xx} + x^2 u_{yy})\right] \end{aligned}$$

Apply the inverse Natural transform on both sides, we get

$$u(x, y, t) = y^2 + \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}\left[\frac{1}{2}(y^2 u_{xx} + x^2 u_{yy})\right]\right]$$

Now applying the Adomian decomposition method

$$\sum_{n=0}^{\infty} u_n(x, y, t) = y^2 + \frac{1}{2} \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}\left[y^2 \sum_{n=0}^{\infty} (A_n)_{xx} + x^2 \sum_{n=0}^{\infty} (A_n)_{yy}\right]\right] \quad (3.4)$$

this gives us

$$\begin{aligned} u_0(x, y, t) &= y^2 \\ u_{k+1}(x, y, t) &= \frac{1}{2} \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}\left[y^2 \sum_{n=0}^{\infty} (A_n)_{xx} + x^2 \sum_{n=0}^{\infty} (A_n)_{yy}\right]\right] \end{aligned}$$

From this recursive relation we can find the series solution in the following way.

$$u_1(x, y, t) = \frac{1}{2} \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}\left[y^2 \sum_{n=0}^{\infty} (A_0)_{xx} + x^2 \sum_{n=0}^{\infty} (A_0)_{yy}\right]\right]$$

Here $F(u) = u$ so that $A_0 = u_0 = y^2$

$$u_1(x, y, t) = x^2 \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

Similarly we can find $u_2(x, y, t)$ as

$$u_2(x, y, t) = y^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

and so on...

Thus the approximate solution of the given two-dimensional fractional heat-like equation is given by(?)

$$\begin{aligned} u(x, y, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\ &= y^2 + x^2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + y^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + x^3 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \\ &= x^2 \sum_{k=0}^{\alpha} \frac{t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha + 1)} + y^2 \sum_{k=0}^{\alpha} \frac{t^{(2k)\alpha}}{\Gamma((2k)\alpha + 1)} \end{aligned}$$

Now the functions $Sinh(t^\alpha, \alpha)$ and $Cosh(t^\alpha, \alpha)$ are defined as

$$\begin{aligned} Sinh(t^\alpha, \alpha) &= \frac{E_\alpha(\lambda) - E_\alpha(-\lambda)}{2} \\ Cosh(t^\alpha, \alpha) &= \frac{E_\alpha(\lambda) + E_\alpha(-\lambda)}{2} \end{aligned}$$

If we put $\alpha = 1$ in the approximate solution $u(x, y, t)$, we obtain

$$u(x, y, t) = x^2 Sinh(t) + y^2 Cosh(t).$$

Example: Consider the following three-dimensional fractional heat-like equation

$$D_t^\alpha u(x, y, z, t) = x^4 y^4 z^4 + \frac{1}{36} (x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}) \quad 0 < \alpha \leq 1 \quad (3.5)$$

subject to initial condition

$$u(x, y, z, 0) = 0 \quad (3.6)$$

subject to boundary condition

$$\begin{aligned} u(0, y, z, t) &= u(x, 0, z, t) = u(x, y, 0, t) = 0 \\ u(1, y, z, t) &= y^4 z^4 (\exp(t) - 1) \\ u(x, 1, z, t) &= x^4 z^4 (\exp(t) - 1) \\ u(x, y, 1, t) &= x^4 y^4 (\exp(t) - 1) \end{aligned}$$

Solution: Applying the Natural transform on both sides of given equation and using the initial condition

$$\begin{aligned} \mathbb{N}[D_t^\alpha u(x, y, z, t)] &= \mathbb{N}[x^4 y^4 z^4 + \frac{1}{36} (x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz})] \\ \mathbb{N}[u(x, y, z, t)] &= \frac{u^\alpha}{s^{\alpha+1}} x^4 y^4 z^4 + \frac{1}{36} \frac{u^\alpha}{s^\alpha} \mathbb{N}[(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz})] \end{aligned}$$

Apply the inverse Natural transform on both sides,we get

$$u(x, y, z, t) = x^4 y^4 z^4 \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{1}{36} \mathbb{N}^{-1} \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}[(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz})] \right]$$

Now applying the Adomian decomposition method

$$\sum_{n=0}^{\infty} u_n(x, y, z, t) = x^4 y^4 z^4 \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{1}{36} \mathbb{N}^{-1} \left[\frac{u^\alpha}{s^\alpha} \mathbb{N} \left[x^2 \sum_{n=0}^{\infty} (A_n)_{xx} + y^2 \sum_{n=0}^{\infty} (A_n)_{yy} + z^2 \sum_{n=0}^{\infty} (A_n)_{zz} \right] \right] \quad (3.7)$$

this gives us

$$u_0(x, y, z, t) = x^4 y^4 z^4 \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

$$u_{k+1}(x, y, z, t) = \frac{1}{36} \mathbb{N}^{-1} \left[\frac{u^\alpha}{s^\alpha} \mathbb{N} \left[x^2 \sum_{n=0}^{\infty} (A_k)_{xx} + y^2 \sum_{n=0}^{\infty} (A_k)_{yy} + z^2 \sum_{n=0}^{\infty} (A_k)_{zz} \right] \right]$$

From this recursive relation we have

$$u_1(x, y, z, t) = x^4 y^4 z^4 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, u_2(x, y, z, t) = x^4 y^4 z^4 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \text{ and so on...}$$

Thus the approximate solution to the given three-dimensional fractional heat-like equation is given by

$$u_N(x, y, z, t) = \sum_{n=1}^N \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha + 1)} \quad (3.8)$$

Now as $N \rightarrow \infty$, we get the solution of given equation as

$$u(x, y, z, t) = \sum_{n=0}^{\infty} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha + 1)} - x^4 y^4 z^4$$

$$= x^4 y^4 z^4 (E_\alpha(t^\alpha) - 1)$$

Where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function.

In case of $\alpha = 1$, we have

$$u(x, y, z, t) = x^4 y^4 z^4 (\exp(t) - 1) \quad (3.9)$$

Example: Consider the following one-dimensional fractional heat-like equation

$$D_t^\alpha u(x, t) = \frac{1}{2} x^2 u_{xx} \quad 0 < \alpha \leq 1, 0 < x < 1, t > 0 \quad (3.10)$$

subject to initial condition

$$u(x, 0) = x^2 \quad (3.11)$$

Solution: Applying the Natural transform on both sides of given equation and using the initial condition

$$\begin{aligned}\mathbb{N}[D_t^\alpha u(x, t)] &= \mathbb{N}\left[\frac{1}{2}x^2 u_{xx}\right] \\ \mathbb{N}[u(x, t)] &= \frac{1}{s}x^2 + \frac{u^\alpha}{s^\alpha} \frac{1}{2}x^2 \mathbb{N}[(u_{xx})]\end{aligned}$$

Apply the inverse Natural transform on both sides, we get

$$u(x, t) = x^2 + \frac{1}{2}x^2 \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}[(u_{xx})]\right]$$

Now applying the Adomian decomposition method

$$\sum_{n=0}^{\infty} u_n(x, t) = x^2 + \frac{1}{2}x^2 \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}\left[\sum_{n=0}^{\infty} (A_n)_{xx}\right]\right] \quad (3.12)$$

this gives us

$$\begin{aligned}u_0(x, t) &= x^2 \\ u_{k+1}(x, t) &= \frac{1}{2}x^2 \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}\left[\sum_{n=0}^{\infty} (A_k)_{xx}\right]\right]\end{aligned}$$

From this recursive relation we have

$$\begin{aligned}u_1(x, t) &= \frac{1}{2}x^2 \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha} \mathbb{N}\left[\sum_{n=0}^{\infty} (A_0)_{xx}\right]\right] \\ &= x^2 \frac{t^\alpha}{\Gamma(\alpha + 1)}\end{aligned}$$

Similarly we can find the other iterative solutions as

$$\begin{aligned}u_2(x, t) &= x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \\ u_3(x, t) &= x^2 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}\end{aligned}$$

and so on...

Thus the approximate solution to the given one-dimensional fractional heat-like equation is given by

$$\begin{aligned}u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\ &= x^2 + x^2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + x^2 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \\ &= x^2 E_\alpha(t^\alpha)\end{aligned}$$

For $\alpha = 1$, we have solution of given equation of the form

$$u(x, t) = x^2 \exp(t) \quad (3.13)$$

4 Conclusion

The main purpose of this article is to combine the two powerful methods namely Natural transform and Adomian decomposition method to solve the fractional heat-like equations. The advantage of this method is its rapid convergence to the exact solution without having tedious numerical computations. This method can solve linear as well as non-linear fractional heat-like equations.

References

- Asiru M. A 2011. Sumudu transform and solution of integral equations of convolution type, *Internat. J. Math. Ed. Sci. Tech.* **32**: 906–910.
- Khan Z.H., Khan W.A. 2011. N-transform properties and applications, *NUST Journal of Engineering Sciences* **1(1)**: 127–133.
- Belgacem FBM, Silambarasan R 2011. Applications of the Natural transform to Maxwell's Equations, *PIERS Suzhou, China* : 899-902.
- Belgacem FBM, Silambarasan R 2012. Theory of the Natural transform, *Mathematics in Engineering Science and Aerospace(MESA)journal* **3(1)**: 99-124.
- Deshna Looner and P.K. Banerji 2013. Natural transform for distribution and Boehmian space, *Mathematics in Engineering, Science and Aerospace MESA* **4(1)**: 69-76.
- Deshna Looner and P.K. Banerji 2013. Application of Natural transform to differential equations, *Journal Indian Academy of Mathematics* **35(1)**: 151-158.
- Chindhe A.D., Kiwne S.B. 2016. Application of Combine Natural Transform and Adomian Decomposition Method in Volterra Integro-Differential Equations, *Mathematical Journal of Interdisciplinary Sciences* **5(1)**: 1–14.
- Chindhe A.D., Kiwne S.B. 2017. Natural Transform and Special Functions, *Journal of New Theory* **15**: 39-47.
- I. Podlubny 1999. Fractional Differential Equations, *Academic Press San Diego*
- K.B. Oldham, J. Spanier 1974. The Fractional Calculus, *Academic Press New York*

- K.S.Miller,B.Ross. 1993. An Introduction to the Fractional Calculus and Fractional Differential Equation,*John Wiley and Sons,Inc.,New York*
- Schneider, W., Wyss, W. 1989. Fractional diffusion and wave equations,*J. Math. Phys.***30**: 134-144.
- Z. Odibat, S. Momani. 2007. Numerical comparison of methods for solving linear differential equations of fractional order,*Chaos. Soliton. Fract.* **31 (5)**: 1248-1255.
- S. Momani, Z. Odibat. 2007. Homotopy perturbation method for non-linear partial differential equations of fractional order.,*Phys. Lett. A***365 (5-6)**: 345-350.
- Q. Wang. 2007. Homotopy perturbation method for fractional KdV equation,*Appl. Math. Comput.***190**: 1795-1802.
- Q. Wang. 2008. Homotopy perturbation method for fractional KdV-Burgers equation. ,*Chaos. Soliton. Fractal.***35**: 843-850.
- J.H. He 1999. Homotopy perturbation technique ,*Comput. Methods Appl. Mech. Eng.***178**: 257-262.
- J.K. Zhou 1986. Differential Transformation and Its Applications for Electrical Circuits,*Huazhong University Press, Wuhan, China*
- A. Al-rabtah, V.S. Ertürk. Momani, S. 2010. Solutions of a fractional oscillator by using differential transform method ,*Computers and Mathematics with Applications***59(3)**: 1356-1362.
- Y. Khan, Q. B. Wu. 2011. Homotopy perturbation transform method for non-linear equations using He's polynomial ,*Computers and Mathematics with Applications***61**: 1963-1967.
- Ganji D. D. 2006. The applications of He's homotopy perturbation method to nonlinear equation arising in heat transfer ,*Physics Letters A***335**: 337–341.
- V. Daftardar-Gieji, H. Jafari 2007. Solving a multi order fractional differential equations using Adomian decomposition ,*J. Math. Anal. Appl.***189**: 541-548.
- G. Jumarie 2009. Laplace's transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative,*Applied Mathematics Letters***22(11)**: 1659– 1664.
- E.Yusufoglu 2006. Numerical solution of Duffing equation by the Laplace decomposition method ,*Applied Mathematics and Computation***177**: 572-580.
- G.Adomain 1994. Solving Frontier problem of Physics: The Decomposition Method ,*Kluwer Acad.Publ.Boston*
- M.Caputo 1997. Linear Models of dissipation whose Q is almost frequency independent Part III. ,*J.Roy.Astr.Soc.***13**: 529-539.