



FIXED POINT THEOREM IN GENERALISED ULTRAMETRIC SPACES

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Abstract: We have introduced here the notion of generalised ultrametric spaces and proved a fixed point theorem.

Keywords: Fixed point theorem, Generalised ultrametric space, Real number, Mapping, Zorn's lemma.

1. Introduction

Branciari introduced the notion of generalised metric spaces by replacing the triangle inequality involving four points instead of three (Branciari 2000). Gajic proved a fixed point theorem in ultrametric spaces (Gajic 2001).

A generalised ultrametric on a non-empty set X is a non-negative real function d on $X \times X$ such that for $x, y \in X$ and for all distinct points $z, w \in X$ we have (van Roaij 1978)

- (1) $d(x, y) = 0$ iff $x = y$,
- (2) $d(x, y) = d(y, x)$ and
- (3) $d(x, y) \leq \max \{d(x, z), d(z, w), d(w, y)\}$.

If d is a generalised ultrametric on X , then (X, d) is called a generalised ultrametric space.

We now give examples of generalised ultrametric spaces which are not ultrametric space.

1.1 Examples

Let $X = \{\alpha, \beta, \gamma, \delta\}$ and the non-negative real valued function d on $X \times X$ be defined as follows:

$$\begin{aligned} d(\alpha, \beta) &= d(\alpha, \gamma) = d(\beta, \delta) = d(\gamma, \delta) = 2 \\ &= d(\beta, \alpha) = d(\gamma, \alpha) = d(\delta, \beta) = d(\delta, \gamma), \\ d(\alpha, \gamma) &= d(\beta, \delta) = 1 \\ &= d(\gamma, \delta) = d(\delta, \gamma) \end{aligned}$$

and $d(x, x) = 0$

for all $x \in X$.

Then d is a generalised ultrametric but not an ultrametric on X .

Let us furnish another example.

If R is the set of real numbers and if the function d is defined on $R \times R$ by $d(x, x) = 0$ for all $x \in R$, $d(0, x) = d(x, 0) = 2^{-|x|}$ for all $x \neq 0$ and $d(x, y) = 1$ for all x, y with $x \neq 0, y \neq 0$ and $x \neq y$, then (R, d) is not an ultrametric space. But it is a generalised ultrametric space.

1.2 Definition

Let (X, d) be a generalised ultrametric space, r a non-negative real number and a & b two distinct points of X with $d(a, b) \leq r$. Then the set $S = \{x \in X \mid \max \{d(a, x), d(b, x)\} \leq r\}$ is called a closed (2c)-sphere having centres at a & b and radius r . The space (X, d) is said to be (2c)-spherically complete if for every decreasing sequence $S_1 \supset S_2 \supset S_3 \supset \dots$ of closed (2c)-spheres the intersection $\bigcap_n S_n$ is nonempty.

2. FIXED POINT THEOREM

Let (X, d) be a (2c)-spherically complete generalised ultrametric space and

$F : X \rightarrow X$ a mapping satisfying the following conditions:

- (a) $d(Fx, Fy) < \max \{d(x, Fx), d(x, y), d(y, Fy)\}$ for all $x, y \in X$ with $x \neq y$ and
- (b) $FFx = x$ if x is not a fixed point of F .

Then F has a unique fixed point.

Proof: Firstly, we prove that F has a fixed point if $FFx = x$. Let S_k denotes the closed (2c)-sphere having centers at x & Fx and indices $d(x, Fx)$.

We write

$$A = \{S_x \mid x \in X\}$$

Now, $S_x \leq S_{x'}$ iff $S_{x'} \subset S_x$ in a partial order. Let us consider a sequence $\{S_{x_n}\}$ of closed (2c)-spheres in A such that $S_{x_{n+1}} \subset S_{x_n}$. Since (X, d) is (2c)-spherically complete, we have

$$\bigcap_n S_{x_n} = S = \phi$$

Let $Y \in S$ and $t \in S \ominus \{y, Fy\}$. We show that for all n

$$\max \{d(t, y), d(t, Fy)\} < d(x_n, Fx_n) \quad \dots\dots\dots (i)$$

If $x_{n_0} = y$ i.e. $Fx_{n_0} = Fy$ for some n_0 , then (i) is obviously true for all $n = n_0$. Therefore, we may assume that $x_n \neq y$ i.e. $Fx_n \neq Fy$. Also we suppose that $x_n \neq Fx_n$ since, otherwise, x_n would be a fixed point. Therefore, by condition (3) of generalised ultrametric space, as mentioned above, we get

$$\begin{aligned} \max \{d(t, y), d(t, Fy)\} &\leq d(y, Fy) \\ &\leq \max \{d(y, x_n), d(x_n, Fx_n), d(Fx_n, Fy)\} \\ &\leq \max \{d(x_n, Fx_n), d(Fx_n, Fy)\} \end{aligned}$$

So, (i) follows if $d(Fx_n, Fy) \leq d(x_n, Fx_n)$.

But if $d(Fx_n, Fy) > d(x_n, Fx_n)$, then

$$\begin{aligned} \max \{d(t, y), d(t, Fy)\} &\leq d(y, Fy) \\ &< d(Fx_n, Fy) \\ &\leq \max \{d(x_n, Fx_n), d(x_n, y), d(y, Fy)\} \\ &= d(x_n, Fx_n) \end{aligned}$$

Thus (i) is true.

It also follows that

$$d(y, Fy) \leq d(x_n, Fx_n) \quad \dots\dots\dots (ii)$$

Next we show that for all n

$$\max \{d(t, x_n), d(t, Fx_n)\} \leq d(x_n, Fx_n) \quad \dots\dots\dots (iii)$$

If $x = y$, then (iii) is obtained from (i). So, we may assume $x_n \neq y$. Again, if $Fx_n = Fy$, then $Fx_n = y$ since otherwise $Fy = y$. Now,

$$d(t, x_n) \leq \max \{d(t, Fx_n), d(Fx_n, y), d(y, x_n)\}$$

$$\begin{aligned}
&= \max \{d(t, Fy), d(Fx_n, y), d(y, x_n)\} \\
&\leq \max \{d(x_n, Fx_n), d(Fx_n, y), d(y, x_n)\} \quad [\text{by (i)}] \\
&\leq d(x_n, Fx_n) \quad \left[\begin{array}{l} \forall y \in S_{x_n} \end{array} \right]
\end{aligned}$$

Thus it follows that (iii) is true in this case also.

So, we can assume both x, y and $Fx_n \bar{F}y$. Then

$$\begin{aligned}
d(t, x_n) &\leq \max \{d(t, Fy), d(Fy, Fx_n), d(x_n, Fx_n)\} \\
&\leq \max \{d(Fy, Fx_n), d(x_n, Fx_n)\} \quad [\text{by (i)}] \\
&\leq \max \{d(y, Fy), d(y, x_n), d(x_n, Fx_n)\} \\
&\leq \max \{d(y, Fy), d(x_n, Fx_n)\} \\
&\leq d(x_n, Fx_n) \quad [\text{by (ii)}]
\end{aligned}$$

$$\begin{aligned}
\text{and } d(t, Fx_n) &\leq \max \{d(t, y), d(y, x_n), d(x_n, Fx_n)\} \\
&\leq \max \{d(t, y), d(x_n, Fx_n)\} \\
&= d(x_n, Fx_n) \quad [\text{by (i)}]
\end{aligned}$$

Therefore (iii) is true and so $t \in S_{x_n}$.

Again, if $y \bar{x}_n$

$$\begin{aligned}
d(Fy, Fx) &\leq \max \{d(y, Fy), d(y, x_n), d(x_n, Fx_n)\} \\
&\leq d(x_n, Fx_n) \quad [\text{by (ii)}] \dots\dots\dots (\text{iv})
\end{aligned}$$

If $Fy \bar{F}x_n$, then

$$d(x_n, Fy) \leq \max \{d(x_n, FFx_n), d(Fx_n, FFx_n), d(Fy, Fx_n)\} \dots\dots\dots (\text{v})$$

For any point $x \in X$ we have

$$d(x_n, FFx) \leq \max \{d(x, Fx), d(Fx, FFFx), d(FFx, FFFx)\} \dots\dots\dots (\text{vi})$$

Using condition (a) of the present theorem we get

$$\begin{aligned}
d(FFx, FFFx) &< \max \{d(Fx, FFx), d(Fx, FFFx), d(FFx, FFFx)\} \\
&\Rightarrow d(FFx, FFFx) < d(Fx, FFx) \quad \dots\dots\dots (\text{vii})
\end{aligned}$$

$$\text{Similarly, } d(Fx, FFx) < d(x, Fx) \quad \dots\dots\dots (\text{viii})$$

$$4 d(FFx, FFFx) < d(x, Fx) \quad [\text{by (ii)}] \dots\dots\dots (\text{ix})$$

$$\begin{aligned}
\text{Also } d(Fx, FFFx) &< \max \{d(x, Fx), d(x, FFx), d(FFx, FFFx)\} \\
&< \max \{d(x, Fx), d(x, FFx)\} \quad [\text{by (ix)}] \dots\dots\dots (\text{x})
\end{aligned}$$

By (vi), (ix) and (x) we get

$$d(x, FFx) < d(x, Fx) \quad \dots\dots\dots (\text{xi})$$

(iv), (v), (viii) and (xi) imply that if $x = x_n$ and $y \bar{x}$, i.e. $Fy \bar{F}x_n$, then

$$d(Fy, x_n) < d(x_n, Fx_n) \quad \dots\dots\dots (\text{xii})$$

By (iv) and (xii) we can say that $Fy \in S_{x_n}$, if $y \bar{x}_n$ and $y \bar{F}x_n$. If $y \bar{x}_n$ i.e. $Fy = Fx_n$, then it is obvious that $Fy \in S_{x_n}$. Also $y \in S_{x_n}$. Thus it follows that $S_y \subset S_{x_n}$. Hence, S_y is an upper bound of the chain $\{S_{x_n}\}$. By Zorn's lemma A has a maximal element S_z (say). As $z \in X$, we show that z is a fixed point of F . Let us suppose $z \bar{F}z$, if possible. Let $\xi \in S_{Fz} - \{FFz\}$. Then

$$\begin{aligned}
d(\xi, z) &\leq \max \{d(\xi, FFz), d(Fz, FFz), d(z, Fz)\} \\
&\leq \max \{d(Fz, FFz), d(z, Fz)\} \\
&< d(z, Fz) \quad [\text{by (vii) with } x = z]
\end{aligned}$$

Also $d(\ell, Fz) \leq d(Fz, FFz) < d(z, Fz)$

Thus $\ell \notin S$. Also by (viii) and (xi), with $x = z$, we get $FFz \in S$. Hence, $S_{Fz} = S_z$. By (viii), with $x = z$, we have $S_{Fz} \notin S_z$. So, $S_{Fz} \neq S_z$. This contradicts the fact that S_z is a maximal element of A .

Hence, $z = Fz$

The uniqueness of the fixed point follows from condition (a).

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