IJRAR.ORG E-ISSN: 2348-1269, P-ISSN: 2349-5138

INTERNATIONAL JOURNAL OF RESEARCH AND ANALYTICAL REVIEWS (IJRAR) | IJRAR.ORG

An International Open Access, Peer-reviewed, Refereed Journal

Invariant relativistic theory of classical and quantum ideal gases

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Abstract. The work presents an invariant relativistic theory of classical and quantum ideal gases subject to the relativistic distributions of Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein. Without applying the Gibbs canonical distribution method and the principle of maximum entropy, an alternative theory of ideal gases has been developed based on finding the macroscopic characteristics of classical and quantum ideal gases (particle number density, pressure and average energy) by statistical averaging over relativistic - invariant velocity distributions (for Maxwell-Boltzmann gas) and moments (for the Fermi-Dirac and Bose-Einstein gas).

For the first time, expressions have been found for the average and root-mean-square particle velocities, which are valid for any ratio of thermal energy and rest energy of particles. For the first time, the limiting velocity of particle flows of quantum relativistic ideal gases has been determined. It has also been established that all real-life relativistic quantum ideal gases have adiabatic exponents that satisfy a certain fundamental inequality.

I. Invariant relativistic theory of classical ideal gas

Annotation. The purpose of this study is to develop an original theory of a relativistic ideal gas and to prove the validity of the postulate of the special theory of relativity for the characteristic (i.e., arithmetic mean, root-mean-square) velocities of particles of a relativistic ideal gas even in the massless limit. In this work, the following original methods are used for the first time in the theory of a relativistic ideal gas: the method of nonlinear transformation to prove of the distribution function to find the distribution function of the velocities of particles of a relativistic ideal gas; the equation of state of a relativistic ideal gas was first obtained by averaging the relativistic - invariant components of the energy - momentum tensor of a system of noninteracting particles, i.e. ideal gas by the distribution function of the velocities of their particles. The uniqueness and definiteness of the distribution function of the velocities of the particles of a relativistic ideal gas are proved on the basis of the well-known relativistic invariance of the distribution function. For the first time, expressions were obtained for the arithmetic mean and mean square velocities of particles of a relativistic ideal gas. For the first time, a fundamental conclusion is made about the validity of the postulates of the special theory of relativity for the characteristic velocities of particles of a relativistic ideal gas. An equation of state for a relativistic ideal gas is obtained, which relates its pressure, average energy density and temperature.

Keywords: distribution function, relativistic ideal gas, arithmetic mean and mean square velocity, equation of state, massless limit.

Introduction

In the last twenty years, especially on the eve of the celebration of the centenary of the creation of the special theory of relativity by the great A. Einstein, interest in the problems of relativistic statistical physics has sharply increased (see, eg, [1,2] and the literature cited in these works). Naturally, this is due to the need to solve a number of problems in plasma physics [3], relativistic kinetic theory [4,5], super - nonequilibrium relativistic thermodynamics in subatomic physics [6].

It should be noted that such questions arose immediately after the construction of the special theory of relativity. Planck and other classics of physics noted that the Maxwellian distribution of velocities contradicts the fundamental postulate of relativity - according to which the speed of particles is limited by the speed of light in emptiness - the limiting speed in nature [7].

The history of generalization of the Maxwellian velocity distribution for the relativistic case lasts more than a hundred years [2,5,7-11]. As the analysis of the literature on this topic shows, the first work in this direction was published by F. Juttner [12]. Despite the fact that the Juttner velocity distribution was the correct relativistic generalization of the Maxwellian velocity distribution (which we will prove below), there is still a serious debate about its correctness and the search for various "modifications" of the Juttner velocity distribution continues. Even in the work of recent years [2, 7], continue to obtain, in the language of their authors, curiously surprising results in this regard.

Therefore, we have to admit (although this seems unlikely) that the problem formulated at the beginning of the twentieth century remains unsolved. Those no work has proved the validity of the fundamental postulate of the special theory of relativity for the velocities of particles of a relativistic ideal gas. It is clear that for this it is necessary to find the distribution function of the particle velocity of a relativistic ideal gas and, on its basis, to determine the expressions for the characteristic velocities of the particles of a relativistic ideal gas. And then, by passing to the limit, prove that even in the massless limit, these speeds do not exceed the limiting speed in nature.

Another important issue of relativistic statistical physics, as is known, is the derivation of the equation of state for a relativistic ideal gas [7-11] - which establishes a relationship between the pressure, average energy density and temperature of a relativistic ideal gas. In the existing literature, the equation of state for a relativistic ideal gas is obtained thermodynamically [1,4,5,7-12], namely, on the basis of a relativistic generalization of the expression for the free energy or the Gibbs partition function (through which all thermodynamic quantities describing the states of an ideal gas [13, 14]). In our work, we present a new derivation of the equation of state for a relativistic ideal gas by an original method, namely, on the basis of averaging the temporal and spatial components of the energy tensor - momentum of a system of noninteracting relativistic (structureless) particles over the velocity distribution function of particles of a relativistic ideal gas. In the context of the problem under consideration, this method seems to us more expedient - a direct method and corresponds to the spirit of Maxwell's own works [15]. Consequently, the proposed theory is a relativistic generalization of the Maxwellian theory of an ideal gas and goes over to the latter in the nonrelativistic limit.

In accordance with the above, the article is organized as follows. In the first section, the distribution function of the particle velocity of a relativistic ideal gas is found using the method of transformation. In the second section, formulas for the mean and root-mean-square velocity of particles of a relativistic ideal gas are obtained. The form of the distribution function of the particle velocity of a relativistic ideal gas in the ultra - relativistic limit is also found here in the case of massless particles (massless particl's). The third section is devoted to a new derivation of the equation of state for a relativistic ideal gas - by averaging macroscopic quantities over the distribution function of the particle velocity of a relativistic ideal gas. In the conclusion, the main conclusions of the proposed theory are presented, as well as their possible applications in high energy physics and relativistic cosmology.

1.1. The distribution function of the particle velocity of a relativistic ideal gas

We will begin the presentation of the proposed theory of a relativistic ideal gas by finding the distribution function of the particle velocity of this gas, since it plays a key role in any statistical system [13-15] and, in particular, in the theory of an ideal gas of relativistic particles [1-12]. Figuratively speaking, the distribution function is the cornerstone of kinetic theory.

Note that the desired velocity distribution function can be found in various ways: as a stationary equilibrium solution of the relativistic kinetic Boltzmann equation [2, 4, 5]; based on the principle of maximum entropy [8-11]. Without discussing here the methods used (they are discussed in more detail in the above literature) and without diminishing the value of these works, we propose the simplest way to determine the velocity distribution function - using the method of transforming the distribution function (see, for example, [18]).

According to this method, on the basis of the known distribution function of the moment of the particles of a relativistic ideal gas $f(\vec{\mathrm{P}}) = f(\mathrm{P}_x, \mathrm{P}_y, \mathrm{P}_z)$, we must determine the distribution function of the velocity of particles of a relativistic ideal gas $\varphi(\vec{v}) = \varphi(v_x, v_y, v_z)$. Here: P_x, P_y, P_z and v_x, v_y, v_z are the components of moments and velocities of particles along the corresponding axes x, y, z. Then, based on the method of transformation, we easily obtain the following relation. Using the transition from the moment of particles to their velocities, we obtain:

$$\int f(P_x, P_y, P_z) dP_x dP_y dP_z = \int \varphi(v_x, v_y, v_z) |D| dv_x dv_y dv_z . \tag{1}$$

Here D is the transformation determinant defined by the following expression

$$D = \begin{vmatrix} \dfrac{\partial P_x}{\partial v_x} & \dfrac{\partial P_x}{\partial v_y} & \dfrac{\partial P_x}{\partial v_z} \\ \dfrac{\partial P_y}{\partial v_x} & \dfrac{\partial P_y}{\partial v_y} & \dfrac{\partial P_y}{\partial v_z} \\ \dfrac{\partial P_z}{\partial v_x} & \dfrac{\partial P_z}{\partial v_y} & \dfrac{\partial P_z}{\partial v_z} \end{vmatrix} . \tag{2}$$

Note that the determinant of the transformation $D \equiv \frac{\partial(P_x,P_y,P_z)}{\partial(v_x,v_y,v_z)}$ will be unique and the same for any type of DF$f(\vec{P})$(the form of which we specify below).

Here: P_x, P_y, P_z is defined according to the special theory of relativity by the following expressions:

$$\vec{P} = P_x\vec{i} + P_y\vec{j} + P_z\vec{k} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}. \tag{3}$$

Now, taking into account the expression D_1, D_2 and D_3 (9), we obtain that $D_1 - D_2 + D_3 = \gamma^2$. The determinant of the transformation D from pulses P_x, P_y, P_z to velocities v_x, v_y, v_z is finally determined by the following expression [2, 24, 25]:

$$D = m^3\gamma^5 = m^3\left(1-\frac{v^2}{c^2}\right)^{-\frac{5}{2}}. \tag{4}$$

Note that the found determinant of the transformation D from momenta to velocity remains valid for an arbitrary distribution function of momenta of particles of a relativistic ideal gas. In particular, for quantum relativistic ideal gases, the particles of which obey, as is well known to the statistics of Fermi-Dirac and Bose-Einstein [19].

Before proceeding to the definition of the distribution function of the velocities of the particles of a relativistic ideal gas, I would like to bring the following transformation of the elementary volume from the space of momenta to the space of velocities (which follows from (1):

$$dP_x dP_y dP_z = D dv_x dv_y dv_z \tag{5}$$

Hence, in particular, it follows that

$$P^2 dP = Dv^2 dv \tag{6}$$

which is easy to verify, given that

$$P = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \frac{dP}{dv} = \frac{m}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}. \tag{7}$$

Now let us determine the distribution function of the velocity of particles of a relativistic ideal gas using the Boltzmann distribution for the momenta of particles of a relativistic ideal gas [18]:

$$f(\vec{p}) = f(P_x, P_y, P_z) = B\exp\left[-\frac{\sqrt{E_0^2 + p^2c^2}}{kT}\right]. \tag{8}$$

where $E_0 = mc^2$is the rest energy of a gas particle, kT is thermal energy, B is a constant, which is determined, as always, by the normalization condition for the distribution function, will be found below.

Further, taking into account that the momentum of a relativistic particle $\vec{P}$ is related to its velocity $\vec{v}$ in a nonlinear manner, according to expression (3)

$$E = \sqrt{E_0^2 + p^2c^2} = mc^2\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \tag{9}$$

based on the DF of the moments (8) $f(\vec{P})$ (14), we find the distribution function of the velocity vector of the particles of the relativistic ideal gas

$$\varphi(v_x, v_y, v_z) = B\exp\left[-b\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right], \quad b = \frac{mc^2}{kT} \tag{10}$$

Therefore, the probability that the speed lies in the interval $(\vec{v}, \vec{v} + d\vec{v})$ or that, the same - the components of the velocity in the interval $(v_x, v_x + dv_x)$, $(v_y, v_y + dv_y)$ and $(v_z, v_z + dv_z)$ is determined according to (10) as follows

$$dW = \varphi(v_x, v_y, v_z) dv_x dv_y dv_z. \tag{11}$$

Finally, in the conclusion of the section, we find the distribution function of the modulus of particle velocities based on the formula for transforming the distribution function from $f(\vec{p})$ to $f(\vec{v})$ (1) and formula (6), taking into account expressions (3) and (10):

$$\int f(\vec{p})d\vec{p} = \int \varphi(\vec{v})d\vec{v} = B \cdot 4\pi \int_0^c m^3 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}} \cdot$$
$$\cdot v^2 \exp\left[-b\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] dv \equiv \int_0^c F(v)dv. \tag{12}$$

Thus, the distribution function of the velocity modulus of particles of a relativistic ideal gas is determined by the following expression

$$F(v) = 4\pi m^3 v^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}} \exp\left[-b\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] \cdot B. \tag{13}$$

Further, introducing the normalized particle velocity $u = \frac{v}{c}$, we can find on the basis of (13) the reduced distribution function of the modulus of the normalized particle velocity $F(u)$

$$\int_0^c F(v)dv = B \cdot 4\pi(mc)^3 \int_0^1 F(u)du. \tag{14}$$

Here

$$F(u) = u^2\left(1 - u^2\right)^{-\frac{5}{2}} \exp\left[-b\left(1 - u^2\right)^{-\frac{1}{2}}\right]. \tag{15}$$

Now we find the constant B using the normalization condition for the distribution function of the velocity moduli of particles of a relativistic ideal gas $F(v)$

$$\int_0^c F(v)dv = B \cdot 4\pi(mc)^3 \int_0^1 F(u)du = 1. \tag{16}$$

Therefore, the constant B according to (14) - (16) is determined by the expression

$$B = \frac{1}{4\pi(mc)^3} \cdot \frac{b}{k_2(b)}.$$

Thus, the reduced distribution function of the moduli of the normalized velocities of particles of a relativistic ideal gas can be written in the following compact form:

$$\Phi(u) = \frac{bF(u)}{k_2(b)} \equiv \frac{b}{k_2(b)} u^2\left(1 - u^2\right)^{-\frac{5}{2}} \exp\left[-b\left(1 - u^2\right)^{-\frac{1}{2}}\right]. \tag{17}$$

The distribution function of the velocity modulus of particles of an ideal gas found by us can be used to calculate the cross section for collisions of particles in a relativistic ideal gas [20] and the cross section for reactions of ultrarelativistic particles in subatomic physics [6]. It may be of interest in studies of the relativistic ionization of atoms by high-energy gas particles, as well as in the field of relativistic laser spectroscopy.

At the end of the section, we present a proof of the invariance of the particle velocity distribution functions of a relativistic ideal gas.

The proof given here of the derivation of the velocity distribution function of particles of a relativistic ideal gas MB is final. This follows from the fact that the relativistic Boltzman distribution function for moments is relativistic invariant. Indeed, taking into account that the four-dimensional momentum of gas particles is

$$P^i = mv^i = m\left(\frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

and the four-dimensional velocity of the rest coordinate system (V = 0) is

$$V^i = (c, 0)$$

we get that

$$P_i V^i = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{(mc^2)^2 + P^2c^2}$$

Since the scalar product of any four-dimensional vectors is invariant, any equilibrium distribution possesses this property.

Obviously, the total probability

$$\int f(P_1, P_2, P_3) dP_1 dP_2 dP_3 = 1$$

does not depend on the transformation of the distribution function, which was used to find the relativistic velocity distribution function.

Let us assume that the momentum distribution function of particles of a relativistic ideal gas (8) is invariant if next relation is invariant

$$\frac{i_0}{T_0} = \frac{i}{T}$$

Here $i_0 = i = P_i V^i$ — according to the above, it is an invariant-as a scalar product of the vectors P_i and V^i and, as is known, does not change under Lorentz transformations.

Hence, we get that the condition T_0=T must be fulfilled, i.e. the temperature of the RIG is the same in all Inertial System Frame (ISF).

Thus, the main drawback of the existing non-invariant RIG theories is eliminated, according to which, during Lorentz transformations, both the DF and the macroscopic characteristics of the RIG change – which contradicts the invariance of the laws of nature in all ISF.

Summarizing the above, we will come to the fundamental conclusion that during the transition from one ISF to another, the statistical properties of the RIG do not change: neither the velocity distribution function nor the equation of state of the RIG do not change under Lorentz transformations, i.e. they are invariant in all ISF. Naturally, this also holds true for quantum RIG.

1.2. Characteristic velocities of particles of a relativistic ideal gas

Based on the distribution function of the velocity module of particles of a relativistic ideal gas found in the previous section, it is possible to find any macroscopic characteristic of this gas depending on the velocity based on the formula for calculating the mean

$$< G >= \int_0^C G(v) F(v) dv \tag{18}$$

The quantity included in this expression, as is known, gives the probability of finding the particle velocity modulus in the velocity interval $[v, v + dv]$

$$dW = F(v) dv . \tag{19}$$

Then, using the explicit form of the relativistic velocity distribution $F(v)$ (19) and the definition of the function $\Phi(u)$ (17), we obtain the following expression to find the above mentioned probability

$$dW = B \cdot 4\pi m^3 v^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}} \exp\left[-b\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] dv = \Phi(u) du . \tag{20}$$

Here $\Phi(u)$ is the reduced distribution function of the modulus of the normalized particle velocities determined according to (17), by the formula

$$\Phi(u) = \frac{dW}{dU} = \frac{b}{k_2} u^2 (1 - u^2)^{-\frac{5}{2}} exp\left[-b(1 - u^2)^{-\frac{1}{2}}\right]. \tag{21}$$

However, according to the Maxwell distribution, which is obtained from the relativistic distribution (13) in the limit $kT << mc^2$ and $v << c$, i.e. in the nonrelativistic approximation, the probability that the modulus of the particle's velocity belongs in the velocity range $[v, v + dv]$ is determined by the expression [15]

$$dW = F_M(v) dv = \frac{4}{\sqrt{\pi}} \cdot u^2 \exp(-u^2) du = \Phi_M(u) du . \tag{22}$$

Here u is the particle velocity normalized to the most probable velocity $v_{mp} = \sqrt{\frac{2kT}{m}}$ i.e. $u = \frac{v}{v_{mp}}$. Comparing the relativistic distribution function $\Phi(u)$ with the Maxwellian distribution function $\Phi_M(u)$, we come to the conclusion that, unlike the latter, the first distribution cannot be represented in a single universal form for all gases. The reason for this difference, as can be seen from the comparison of the two distributions, is the presence of the parameter b in the relativistic distribution, which is equal to the ratio of the rest energy of gas particles to the thermal energy kT. In particular, as we will prove below, it is this circumstance that leads to the fact that for a relativistic ideal gas the law of uniform distribution of the average kinetic energy over the degrees of freedom does not hold. Attention was drawn to this, for example, in Pauli's book [21].

The unusualness of the relativistic distribution function (or what, the same thing, the relativistic distribution) $\Phi(u)$ makes us be more attentive to this attractive person (it is clear that such people are extremely rare, if not completely!). Of course, we are primarily interested in, as in the case of the Maxwellian distribution, the characteristic velocities of the relativistic distribution, namely: the most probable, mean and root-mean-square velocities of particles of a relativistic ideal gas.

The most probable particle velocity corresponds to the maximum of the relativistic velocity distribution function $F(v)$ or the relativistic distribution function $\Phi(u)$. It is determined by the extremum condition $\frac{dF(v)}{dv}=0$, which is equivalent to condition $\frac{d\Phi(u)}{du}=0$. Calculating this derivative and making a number of simplifications, we obtain the following equation

$$\left[2+\frac{u^2}{1-u^2}\left(5-\frac{b}{\sqrt{1-u^2}}\right)\right]\Phi(u)=0. \tag{23}$$

Further, taking into account that for the values of the parameter b other than zero: $\Phi(u)=0$ at $u=0$ and $u=1$, we come to the conclusion that these values of the velocities correspond to the zeros of the relativistic distribution.

In addition to these roots, equation (23) also has intermediate roots, as follows from it, determined by the solution of the following equation

$$bu^2=(2+3u^2)\sqrt{1-u^2}. \tag{24}$$

We further restrict ourselves to analyzing its solutions only for very large and very small values of the parameter b [1].

As follows from equation (24), for any finite values of b, its roots belong to the interval of normalized velocities $0<u<1$.

For $u<<1$, which corresponds to nonrelativistic particles, from (24) we obtain that if $b>>1$ or $\kappa T<<mc^2$, it has the following approximate solution

$$u_{mp}=\frac{v_{mp}}{c}=\sqrt{\frac{2}{b}}=\sqrt{\frac{2kT}{mc^2}}. \tag{25}$$

Therefore, as expected, in this nonrelativistic limit we obtain a result following from the Maxwellian distribution of $F_M(v)$ or $\Phi_M(u)$ - which is valid in this case.

Now we find a solution to equation (24) close to the limiting one, i.e. $u=1-\varepsilon\,(0<\varepsilon<<1)$. Then it follows from this equation that

$$u\approx 1-\frac{b^2}{50}. \tag{26}$$

This solution corresponds to very small values of the parameter $b<<1$, which means an ultrarelativistic limit. Further, taking into account the asymptotic behavior of the modified second-order Bessel function $\kappa_2(b)\approx\frac{2}{b^2}$, according to (21), we come to the conclusion that the maximum of DF $\Phi(u)$ is described by the following expression

$$\Phi_{макс}\approx\frac{2}{b^2}. \tag{27}$$

Therefore, in contrast to the Maxwell distribution $\Phi_M(u)$ (22), which has a maximum at $u_{mp}=1$ at any value of b, the relativistic distribution $\Phi(u)$ (21) has a maximum value depending on the parameter b. In particular, at $b<<1$, according to (27), the maximum of this distribution grows, i.e. the relative number of particles grows with velocities close to the limiting $u\approx 1$. However, as is known, the maximum of the Maxwellian distribution $F_M(v)$ decreases with increasing temperature, i.e. tends to zero at very high temperatures. This is the main difference between the relativistic distribution $F(v)$and the Maxwellian distribution $F_M(v)$!

We now turn to determining the average velocity of particles of a relativistic ideal gas. According to the formula for calculating the means (18), we obtain that

$$<v>=\int_0^C vF(v)dv=c\int_0^1 u\Phi(u)du=c<u>. \tag{28}$$

Using expression (28), we obtain the following formula for the average velocity of particles of a relativistic ideal gas [2, 24, 25]:

[1]We will not give general solutions to this equation because of its cumbersomeness.

$$< u >= \frac{2e^{-b}}{k_2(b)}\left(\frac{1+b}{b^2}\right). \qquad (29)$$

In a similar way, we determine the mean square of the velocity again using the formula for calculating the means (18), according to which

$$< v^2 >= \int_0^C v^2 F(v)dv = c^2 \int_0^1 u^2 \Phi(u)du = c^2 < u^2 >. \qquad (30)$$

Using (21) and (30), we obtain the following formula for the mean square of the normalized velocity of particles of a relativistic ideal gas [2, 24, 25]

$$< u^2 >= 1 - \frac{k_1(b) - ki_1(b)}{k_2(b)} b. \qquad (31)$$

The found expressions for the mean modulus and the mean square of the modulus of the normalized particle velocity (38), (39) allow us to determine another most important characteristic of the relativistic distribution (27) - the root-mean-square fluctuation of the velocity of gas particles:

$$Dv = < v^2 > - < v >^2 = c^2[< u^2 > - < u >^2] = c^2 Du, \qquad (32)$$

which describes the characteristic spread of the velocities of particles of a relativistic ideal gas.

Now let us analyze the behavior of the found characteristic velocities for a very large value of the parameter b, i.e. $b >> 1$. In this case, as already noted, the thermal energykT is much less than the rest energy of the particles mc^2. Further, taking into account the asymptotics of $k_1(b)$ and $ki_1(b)$ [22]:

$$k_1(b) \approx \sqrt{\frac{\pi}{2b}} e^{-b}\left(1 + \frac{3}{8b} - \frac{15}{2} \cdot \frac{1}{(8b)^2} + \ldots\right), \qquad (33)$$

$$ki_1(b) \approx \sqrt{\frac{\pi}{2b}} e^{-b}\left(1 - \frac{5}{8b} + \frac{129}{128} \cdot \frac{1}{b^2} + \ldots\right), \qquad (34)$$

in the considered limit, we obtain

$$[k_1(b) - ki_1(b)]b \approx \sqrt{\frac{\pi}{2b}} e^{-b}\left(1 - \frac{9}{8b}\right). \qquad (35)$$

Now recalling the asymptotics of the modified second-order Bessel function $k_2(b)$

$$k_2(b) \approx \sqrt{\frac{\pi}{2b}} e^{-b}\left(1 + \frac{15}{8b} + \ldots\right), \qquad (36)$$

on the basis of expressions (30), (35), (36) we obtain the following formula for the mean square of the velocity of ideal gas particles

$$< v^2 > \approx c^2 \cdot \frac{3}{b}\left(1 - \frac{45}{(8b)^2}\right) \approx \frac{3\kappa T}{m}. \qquad (37)$$

This is a result following from the Maxwellian distribution for the mean square of the particle velocity in the nonrelativistic limit.

In the same limit, in accordance with expressions (28), (29), and (36), we obtain

$$< v > \approx c \cdot 2\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{b}} \approx \sqrt{\frac{8}{\pi} \cdot \frac{\kappa T}{m}}. \qquad (38)$$

This is the formula for the average velocity of the Maxwellian distribution. Thus, within the framework of the applicability of the Maxwellian distribution, the root-mean-square fluctuation of the velocities of particles of an ideal gas according to (32), (37), and (38) is determined by the following expression

$$Dv = c^2 Du \approx c^2\left(3 - \frac{8}{\pi}\right)\frac{1}{b} \approx \left(3 - \frac{8}{\pi}\right)\frac{kT}{m}. \qquad (39)$$

Finally, consider the case of ultrarelativistic (or massless) particles, which corresponds to very small values of the parameter b, i.e. $b << 1$ (or $b \to 0$). Then, taking into account the asymptotics of the functions $k_1(b)$, $k_2(b)$ and $ki_1(b)$ in this limit [22]

$$k_1(b) \approx \frac{1}{b}, \quad k_2(b) \approx \frac{2}{b^2}, \quad ki_1(b) \approx \frac{\pi}{2}, \qquad (40)$$

we obtain from (29) and (31) the following results

$$< u >= 1 - b^2, \quad < u^2 > \approx 1 - \frac{1}{2}b^2. \qquad (41)$$

The latest results mean that as m → 0, these characteristic speeds tend to the speed of light c.

Consequently, in the ultrarelativistic limit, the root-mean-square fluctuation of the velocities of particles of an ideal gas is determined by the following formula

$$Dv = c^2 Du \approx c^2 \frac{3}{2} b^2 \approx \frac{3}{2} c^2 \left(\frac{mc^2}{\kappa T} \right)^2 . \quad (42)$$

Thus, in this section, on the basis of the relativistic distribution of velocities (19), it is proved that the characteristic velocities of gas particles: the most probable, mean and root-mean-square velocities do not exceed the speed of light. In particular, taking into account the unattainability of absolute zero temperature, proved by Nernst, we come to the conclusion that only massless particles at any temperature have speeds equal to the speed of light. In addition, it is possible to show the boundedness of the root-mean-square fluctuations of particle velocities for any value of the parameter b.

Concluding the section, we note that these qualitative considerations in obtaining solution (24) can be easily obtained by finding the values of b corresponding to the most probable velocities $u_{mp} = \frac{v_{mp}}{c}$ which are determined as follows:

$$b = \left(3 + \frac{2}{u^2} \right) \sqrt{1 - u^2} .$$

From here, for $u << 1$ and $u \approx 1 - \varepsilon \quad (0 < \varepsilon << 1)$, we obtain approximate solutions (25) and (26) of equation (24).

Based on the above analyzes, we come to the conclusion that in the massless limit all characteristic particle velocities tend to the speed of light. In other words, the distribution function of the velocity modulus of particles of an ultrarelativistic ideal gas becomes delta-shaped $F_0(\vartheta) = \delta(\vartheta - c)$, and their directions are completely random and equally probable, i.e. distributed isotropically. Indeed, the relativistic distribution function of the normalized velocity *Φ (u)* at *b << 1* takes the following form

$$\Phi(u) \approx \frac{b^3}{2} \cdot u^2 (1 - u^2)^{-\frac{5}{2}} \exp\left[-b(1 - u^2)^{-\frac{1}{2}} \right].$$

Further, setting $u^2 = 1 - \varepsilon^2 \ (\varepsilon \to 0)$, , we obtain

$$\Phi(u) \approx \frac{b^3}{2\varepsilon^5} \exp(-\frac{b}{\varepsilon}).$$

Hence it is clear that if $b = \sqrt{\varepsilon},$ then at $\varepsilon \to 0$ the function *Φ (u)* tends exponentially to zero. If $b = \varepsilon^{\frac{3}{2}}$, then at $b \to 0$ *Φ (u)* becomes infinite. Therefore, in this case, the reduced distribution function will be delta-shaped, i.e.

$\Phi(u) = \delta(1 - u)$ (fig. 1).

Thus, based on the analysis of the expressions for the characteristic velocities of the particles of a relativistic ideal gas, we come to the fundamental conclusion that the most probable average and root-mean-square velocity does not exceed the speed of light - the limiting velocity in nature. This proves the validity of the postulates of the special theory of relativity in relativistic statistical physics.

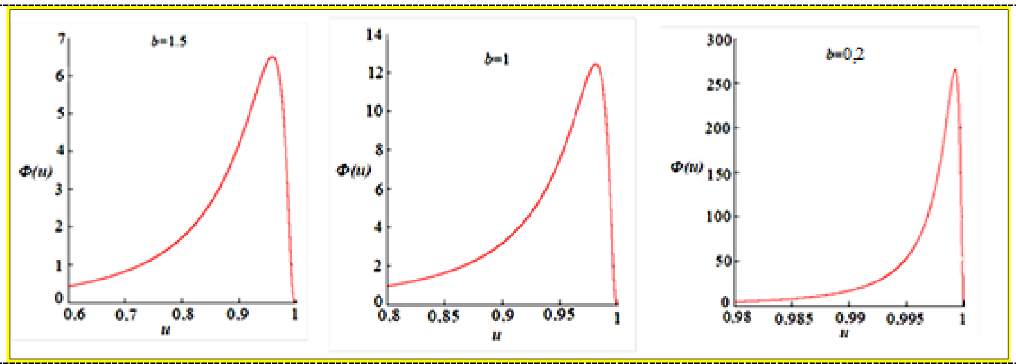

Figure: 1. The distribution function of the moduli of the normalized velocities of the particles of a relativistic ideal gas (b = 1.5; 1; 0.2)

As an example, we find the root-mean-square velocity of the particles of a relativistic ideal gas at the value of the parameter, when the thermal energy is equal to one third of the rest energy of the gas particle.

Using formula (31) and taking into account the values of the special functions appearing in it, we obtain

$$\vartheta_{кв} \equiv \sqrt{<\vartheta^2>} \approx 0.515\, c.$$

However, according to Maxwell's formula for the mean square velocity (37) $\vartheta_{кв} = c$. This result has no physical meaning, since, according to the special theory of relativity, no particle with a nonzero mass can have a speed equal to the speed of light. The latter conclusion also follows from the relativistic distribution function of the particle velocity modulus found by us: if $m \neq 0$, then at $\vartheta \to c$ the function $F(\vartheta) \to 0$.

1.3. Equation of state for a relativistic ideal gas

As is known, the properties of any ideal gas are determined by its equation of state, which connects three thermodynamic quantities: pressure, average energy density and temperature [13,23]. Therefore, our goal in this section is to determine the equation of state for a relativistic ideal gas.

Before proceeding with the direct solution of this problem, we give a short proof of expressions for the average energy density and pressure of a relativistic ideal gas. To do this, we use the following well-known expression for the energy-momentum tensor of a system of noninteracting particles [23]:

$$T^{ik} = nmc < \frac{dx^i}{dt} \cdot \frac{dx^k}{dS} >, \; dS = c\sqrt{1-\frac{v^2}{c^2}}\,dt. \tag{43}$$

where$n$ is the number of particles per unit volume, and the angle brackets denote averaging over the velocity distribution function of gas particles.

Further, taking into account the definitions of the four-dimensional radius vector of the particle x^i

$$x^i = \left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z) = (ct, r^{\rightarrow}), \tag{44}$$

we find the following expressions necessary to find T^{ik}:

$$\frac{dx^i}{dt} = \left(c, v^{\rightarrow}\right) = \left(c, v_x, v_y, v_z\right), \tag{45}$$

$$\frac{dx^k}{dS} = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{v^{\rightarrow}}{c\sqrt{1-\frac{v^2}{c^2}}}\right), \tag{46}$$

where $v^{\rightarrow}$ is the usual three-dimensional particle velocity.

Therefore, using expressions (43) - (46), one can determine any component of the energy-momentum tensor T^{ik} of a system of noninteracting particles, for example,

$$T^{00} = nmc^2 < \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} >, \quad T^{\alpha\alpha} = nm < \frac{v_\alpha^2}{\sqrt{1-\frac{v^2}{c^2}}} >, \tag{47}$$

where $\alpha = x, y, z$.

Hence it follows that the time component of the energy-momentum tensor T^{00} is equal to the average energy density ρ of the relativistic ideal gas, determined by the expression

$$\rho = nmc^2 < \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} >. \tag{48}$$

Now, using the spatial components of the energy-momentum tensor, we obtain that

$$T^{xx} + T^{yy} + T^{zz} = nmc^2 < \frac{v^2}{c^2\sqrt{1-\frac{v^2}{c^2}}} >. \tag{47*}$$

However, for any isotropic distribution of particle velocities (otherwise it cannot be due to the equality of all directions in space)

$$T^{xx} = T^{yy} = T^{zz} = P, \tag{49}$$

Thus, from (47*) and (49) we obtain the following expression for the pressure of a relativistic ideal gas:

$$P = \frac{nmc^2}{3} < \frac{v^2}{c^2\sqrt{1-\frac{v^2}{c^2}}} >. \tag{50}$$

We emphasize that we did not cite here the expressions for the mixed components of the energy-momentum tensor T^{ik} when i and k are not equal, simply because they are all equal to zero. Now using the identity

$$\frac{1}{\sqrt{1-u^2}} = \sqrt{1-u^2} + \frac{u^2}{\sqrt{1-u^2}}, \quad u = \frac{v}{c}, \tag{51}$$

and taking into account expressions (48) and (50), we obtain the following important equality, which establishes a relationship between the pressure P and the average energy density ρ of a relativistic ideal gas

$$\rho - 3P = nmc^2 < \sqrt{1-\frac{v^2}{c^2}} > . \tag{52}$$

Thus, as follows from the expressions for the average energy density ρ (48) and pressure P (50), as well as the equation of state (52) connecting them, they are all determined by statistical averaging based on the relativistic distribution of the velocities of their particles $F(v)$ (13). This is the originality of the method used by us for the first time for finding the indicated macroscopic quantities of a relativistic ideal gas.

Let's start solving this problem by determining the average energy density of a relativistic ideal gas based on expressions (48) and (13):

$$\rho = nmc^2 \int_0^C \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} F(v)dv = nmc^2 \int_0^1 \frac{1}{\sqrt{1-u^2}} \Phi(u)du . \tag{53}$$

Now, applying the substitution $u = thx (0 < x < \infty)$ and taking into account expression $\Phi(u)$ (21), we obtain

$$\rho = nmc^2 \frac{b}{k_2(b)} \left(\int_0^\infty sh^2 x e^{-bchx} dx + \int_0^\infty sh^4 x e^{-bchx} dx \right) \tag{54}$$

The integrals included in (54) are respectively equal [22]

$$I_1 = \frac{k_1(b)}{b}, \quad I_2 = 3\frac{k_2(b)}{b^2}, \tag{55}$$

where $k_1(b)$ and $k_2(b)$ are the modified Bessel functions of the first and second order.

Thus, for the average energy density ρ of a relativistic ideal gas, we obtain the following beautiful expression

$$\rho = nmc^2 \left[\frac{k_1(b)}{k_2(b)} + \frac{3}{b} \right], \quad b = \frac{mc^2}{\kappa T} . \tag{56}$$

Next, we calculate the right-hand side of equality (52)

$$nmc^2 < \sqrt{1-\frac{v^2}{c^2}} > = nmc^2 \frac{b}{k_2(b)} \int_0^\infty sh^2 x e^{-bchx} dx . \tag{57}$$

Now, taking into account the first integral in expression (55), from this we obtain

$$nmc^2 < \sqrt{1-\frac{v^2}{c^2}} > = nmc^2 \frac{k_1(b)}{k_2(b)} . \tag{58}$$

Substituting the found mean in the right-hand side of equality (52), we find the following expression for the pressure of the Maxwell-Boltzmann relativistic ideal gas

$$P = nkT . \tag{59}$$

Of course, this result can also be obtained by direct averaging of the expression for pressure (50). However, now, unlike the nonrelativistic ideal gas theory, the pressure of a relativistic ideal gas will not be directly proportional to the average kinetic energy of gas particles $\langle E_k \rangle$, since, according to (56)

$$\langle E_k \rangle = mc^2 \left[\frac{k_1(b)}{k_2(b)} + \frac{3}{b} - 1 \right],$$

is a nonlinear function of the thermal energy kT and the rest energy of the gas particle mc^2. In particular

$$\frac{\langle E_k \rangle}{kT} = b \cdot \left[\frac{k_1(b)}{k_2(b)} + \frac{3}{b} - 1 \right]$$

Using this formula, by measuring $\langle E_k \rangle$ and T, you can determine the rest energy of gas particles mc^2 in a non-trivial way.

Thus, the equation of state of a relativistic ideal gas connecting its pressure, average energy density and temperature is determined by the following expression

$$\rho - 3P = nmc^2 \frac{k_1(b)}{k_2(b)}, \quad b = \frac{mc^2}{\kappa T}. \tag{60}$$

We emphasize that the above results can be obtained by other statistical methods, which were mentioned at the beginning of the first section.

Let us now investigate the asymptotic behavior of the pressure and average energy density of a relativistic ideal gas at the limiting values of the parameter b.

If the thermal energy kT is negligible compared to the rest energy of the gas particles mc^2, then the parameter $b >> 1$. Further, taking into account the asymptotics of the functions $k_1(b)$ and $k_2(b)$ (36), we have

$$\frac{k_1(b)}{k_2(b)} \approx 1 - \frac{3}{2b} - \frac{45}{64b^2}. \tag{61}$$

Then it follows from (56) that in this limit the mean energy density is determined by the following formula

$$\rho \approx nmc^2 + \frac{3}{2}nkT. \tag{62}$$

Consequently, only for a nonrelativistic ideal gas, the law on the uniform distribution of the average kinetic energy over the degrees of freedom is valid, i.e.

$$< E_k >= \frac{3}{2}\kappa T. \tag{63}$$

In the opposite limit $b << 1$ (which means not only $T \to \infty$, but also $m \to 0$!), The modified Bessel functions of the first and second order have the asymptotics indicated in expression (40). Then from (56) we obtain the following formula for the average energy density of an ultrarelativistic ideal gas (or a relativistic gas of massless particles)

$$\rho \approx 3nkT\left[1 + \frac{1}{6}\left(\frac{mc^2}{\kappa T}\right)^2\right]. \tag{64}$$

In this limit, from (60) follows the so-called limiting equation of state

$$P \approx \frac{\rho}{3}, \tag{65}$$

which corresponds to the highest possible pressure at a given average energy density ρ.

Based on the above analyzes of the equation of state (60) relating pressure, average energy density and temperature of a relativistic ideal gas, we can write the following approximate equations of state:

$$P \approx 0, \quad P \approx \frac{\rho}{3}, \tag{66}$$

usually used in the equations of motion of matter (matter and radiation) in cosmological theories [23].

Here, the first equation of state corresponds to a nonrelativistic gas with an ultralow temperature ($T \to 0$), the second - to an ultrarelativistic gas with a very high temperature ($T \to 0$) or equilibrium radiation. These approximate equations of state used are virtually independent of temperature. Therefore, based on the analysis of the equation of motion of matter, where the equations of state are used, which does not take into account the dependence of pressure and average energy density on temperature, it is impossible, in principle, to determine the change in the temperature of the Universe during its evolution. An additional argument in favor of this conclusion is the fact that the very establishment of the equilibrium Boltzmann distribution and the resulting relativistic distribution of the velocities of gas particles is possible only in statistical equilibrium, with a constant temperature in time.

In conclusion, we emphasize that the pressure and average energy density of a relativistic ideal gas take only non-negative values according to their physical meaning. This conclusion remains valid for quantum relativistic ideal gases, since the expressions for the pressure (50) and average energy density (48) of a relativistic ideal gas, as well as the equation of state (52) connecting them, remain valid in the case of quantum statistics. Only averaging follows, it is carried out over the distribution function of the momenta of quantum relativistic ideal gases [24-26].

Thus, no substance can have negative pressure and negative average energy density. This excludes the possibility of the existence of the supposed "dark" matter with such hypothetical properties.

Conclusion

Let us summarize the consequences of the proposed theory of a relativistic ideal gas, compare them with previously known results, and also discuss issues of their further development.

The presentation will be carried out in accordance with the sequence of the issues considered in the article.

1. The distribution function of the particle velocity of a relativistic ideal gas, obtained by us by transforming the distribution function, which is described by expression (13), agrees with the result of Juttner's work [12], obtained by him on the basis of a relativistic generalization of the Gibbs statistic. The “modified” Juttner distribution of velocities obtained in [7] is incorrect, since, according to the authors themselves, it contains a factor - the reciprocal of the

energy of a relativistic particle. This leads to the erroneous conclusion that in the ultra - relativistic limit the "Modified" Juttner distribution of velocities tends to zero. This contradicts the result (27) and does not correspond to reality (see also a number of curious results given in [7]).

2. The formulas describing the mean and root-mean-square velocity of particles of a relativistic ideal gas were not obtained by either Juttner or his "modifiers". Consequently, for the first time in our work, it was proved on their basis that the characteristic velocities of particles of a relativistic ideal gas are limited - the limiting velocity in nature. This confirms the validity of the fundamental postulate of the special theory of relativity in relativistic statistical physics.

3. The equation of state of a relativistic ideal gas, relating the pressure, average energy density and temperature of a relativistic ideal gas described by expressions (56), (59) and (60), agrees with the results obtained by Juttner [12], as well as other researchers [4, 5, 8-11, 16, 17]. But unlike these works, in which thermodynamic methods were used to obtain the equation of state (based on free energy, the principle of maximum entropy, etc.), here it was obtained by a new method, namely, by averaging macroscopic quantities for a system of relativistic non-interacting particles, i.e., e. for a relativistic ideal gas from the distribution function of the velocity of their particles. As far as we know, no one has previously used such a method for obtaining the equation of state for a relativistic ideal gas.
It is of great interest to generalize the theory of a relativistic ideal gas developed above for reference frames moving with constant acceleration [16], as well as for relativistic plasma of colliding beams [3].

4. Analysis of limit case

A. Substantially Nonrelativistic Ideal Gas $b = \frac{mc^2}{kT} \gg 1$.

In this limit, it should be taken into account that

$$E \approx mc^2 + \frac{P^2}{2m}, k_2(b) = \sqrt{\frac{\pi}{2b}} e^{-b}(1 + \frac{15}{8b}).$$

Then we get

$$f \approx \frac{1}{4\pi}\sqrt{\frac{2}{\pi}} \cdot \frac{1}{(mkT)^{\frac{3}{2}}} \exp(-\frac{P^2}{2mkT}).$$

B. Ultra-relativistic limit.

$b = \frac{mc^2}{kT} \ll 1$ (or b→0). Then, taking into account the asymptotics of the function in this limit $k_2(b)$, we obtain $k_2(b) \approx \frac{2}{b^2}$.

In addition it should be taken into account that

$$E \approx pc.$$

Therefore, we have

$$f(p) \approx \frac{1}{8\pi} \cdot \left(\frac{c}{kT}\right)^3 \cdot \exp(-\frac{pc}{kT})$$

Hence it follows that if $T \to 0, \ f(p) \to \delta(p)$.

C. Of the effective cross section of colliding particles

$$\frac{\sigma}{\sigma_{max}} =< \sqrt{(\overrightarrow{u_1} - \overrightarrow{u_2})^2 - [\overrightarrow{u_1}, \overrightarrow{u_2}]^2} >$$

Here $u_{1,2} = \frac{u_{1,2}}{c}$. $u_{1,2}$ — normalized velocities colliding particles of RIG.

Further, the case is considered when, for example, u_2=0. Then, the previous formula is greatly simplified.

$$\frac{\sigma}{\sigma_{max}} =< u >$$

Consequently, for parameter values significantly less than one, i.e. for the UR particles, for example, at $b=10^{-3}$ (which was possible in experiments on particle collisions at CERN, as well as in experiments in Controlled Nuclear Fussion in France), we obtain according to the formula for the average velocity Maxwellian distribution

$$\langle u \rangle \approx 2\sqrt{\frac{2}{\pi}} \cdot 10^2.$$

This result obviously has no physical meaning, since it means that the speed of particles exceeds the speed of light 100 times.

But at the same time, according to the formula for the average RIG speed, in this limit we get

$$\langle u \rangle = 1 - b^2 = 1 - 10^{-6}$$

Consequently, according to the latest results, the effective cross-section of colliding particles of a relativistic ideal gas cannot exceed the effective cross-section of a stationary particle. I.e., starting from a certain threshold energy, the effective cross-section practically does not depend on the average kinetic energy of colliding particles, which is confirmed by experiments conducted at CERN.

As an application of the theory of a relativistic ideal gas, one can point out its possible application for calculating the cross section for collisions of ultra - relativistic heavy ions in a high energy collider [27], as well as for the relativistic kinetic description of the production of fermions and bosons in cosmology.

Completing the work, we cannot but pay attention to, perhaps - not very pleasant and annoying, strange circumstance that has developed in the history of the theory of a relativistic ideal gas: for more than a century after Juttner's work [12], they were engaged in an essentially meaningless business - they tried "Modify" ... the correct Juttner velocity distribution! Here, as they say, there is nothing to be done - this is the nature of things. But the truth, although it is bitter, is always fair ...

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II. Invariant relativistic theory of quantum ideal gas

Annotation. An original invariant relativistic theory of a quantum ideal gas (QIG) of particles, antiparticles, and photons is presented, in which the macroscopic characteristics and the QIG equation of state are determined for the first time on the basis of quantum averaging over the invariant particle momentum distribution function of a relativistic QIG. The condition for the condensation of a relativistic Bose gas is found. Based on it, a relativistic quantum gas model of the Universe is also proposed. The maximum limiting velocity of the QRIG particle flows is determined. In addition, it has been proved that fermions with zero rest mass cannot exist. Otherwise, it turns out that the gas pressure becomes negative, which is not satisfied by any substance known so far. Therefore, the only particle with zero rest mass is the photon. It also ensures the stability of the gaseous Universe, preventing its expansion in space. The latter is due to the fact that the particle flow rate even in a photon gas, i.e. in a bosonic gas of massless particles, is less than the speed of sound due to the fact that the photon gas at any absolute temperatures other than zero has an adiabatic index $\gamma_{ph} > \frac{4}{3}$, which provides the condition $v < c$. It is shown that the particle flow rate of the RQIG cannot exceed $v_{ph} < v_{s.max}$. Here: v_{ph} and $v_{s.max}$are, respectively, the speed of the flow of photon gas particles and the maximum speed of sound. It has been established that the adiabatic indices of the actually existing RQIG satisfy the fundamental inequality $\gamma \geq \gamma_{ph} > \frac{4}{3}$, which ensures non-negativity of the pressure and subsonic velocity of their particle flows.

Keywords: Relativistic quantum ideal gas (RQIG), macroscopic characteristics and equation of state of RQIG, condition for condensation of a relativistic Bose gas, relativistic quantum gas model of the Universe, maximum boundary velocity of QRIG particle flows, stability of the gaseous Universe.

Introduction

After analyzing the literature of the first two decades of our century, we inevitably come, as one of the "luminaries" of science (whose name does not matter in this case) "to an unsightly picture of the state of the relativistic theory of QIG ...".

This means that there are still a number of fundamental issues that have not yet been resolved. Indisputable striking examples of the latter conclusion are the three (?!) laws of temperature transformation during the transition to a moving inertial frame of reference (IFR) [1-3].

But this point of view, whoever did not express it, contradicts invariance, i.e. immutability of the laws of nature in IFR.

In the context under discussion, this means that the statistical nature of the system should be the same in all IFRs, i.e. the distribution function (DF) of the momenta of QIG particles must be invariant, i.e. the same for all observers moving at a constant speed. Otherwise, due to the “transformation” of the DF, the macroscopic characteristics of the QIG change (n- is the average density of the number of particles, ρ- is the average energy density, P- is the pressure), as well as the equation of state of the QIG, which are determined by averaging on its basis.

The relativistic invariant theory (RIT) of a quantum ideal gas (QIG) presented in this article is, in essence, a logical continuation and completion of the relativistic invariant theory of a classical ideal gas or the so-called Maxwell-Boltzmann ideal gas [4-9]. As was proved in [4], the DF of RIG particles, as well as the absolute temperature of RIG, is invariant, i.e. the same in all IFRs. Consequently, as it should be, the macroscopic characteristics and the RIG equation of state connecting them are invariant, regardless of the method of averaging, i.e. based on classical or quantum statistics.

Note that, at the same time, there is still a relativistic theory of QIG in the so-called absolutely synchronous IFR or in an accompanying (i.e., associated with relic radiation) IFR [10-13]. But this obviously leads to the existence of a "separate" IRF, which violates the equality of the IFR and, naturally, contradicts the invariance of the special theory of relativity (STR). It seems to us not reasonable and not consistent with common sense that "the difference in IFR leads to an observable effect" - a conclusion made by a number of authors. The situation is further aggravated by the fact that the so-called Ito-Stratonovich-Klimontovich-Hangi paradox has not been resolved; there are three different solutions (?!) of the same Fokker-Planck equation for the DF on momentum or DF of velocities [14]. But the issue of violation of the uniqueness of the solution of the Fokker-Planck equation requires a special study and will not be considered here. The situation of continuing “modifications” of the Juttner distribution and work in this direction is discouraging [13-20].

Summarizing the above, we can now formulate the goal of this work: to develop an invariant relativistic theory of the QIG of particles, antiparticles and photons, which takes into account the possibility of the creation and annihilation of particles, and also to determine the velocity of the flow of QRIG particles on its basis.

Based on the formulation of the question, the presented theory will be presented in the following logical sequence: in the first section, the Maxwell-Boltzmann model of the relativistic QIG is considered, which is auxiliary, but very necessary for understanding the theory presented below. In the second section, we obtain general expressions for the macroscopic characteristics and equations of state for the RQIG of Bose and Fermi particles in the form of a summable series. In the third section, the properties of the RQIG of bosons are investigated. In the fourth section, the features of the behavior of the RQIG of fermions are studied. In the fifth section, a formula is defined that allows one to calculate the maximum flow rate of QRIG particles. This section also analyzes the physical consequences arising from the first law of relativistic thermodynamics for adiabatic processes and the laws of conservation of particle flux, as well as the equation of state. In it, in particular, it is proved that no other particles with zero rest mass can exist, except for the only Bose particle, the photon. It is shown that otherwise, i.e. in the presence of a Fermi particle with zero rest mass, we obtain results that have no physical meaning and do not correspond to the equations of state of substances known to physics. The final part of the article summarizes the results obtained, their comparison with the available literature, as well as their possible applications and prospects for their further development.

Macroscopic characteristics of RQIG.

We begin our presentation by generalizing the definition of the macroscopic characteristics of the RQIG to the case of averaging over quantum statistics.

To do this, we proceed from the fact that, according to our previous works [4-9], the average energy density and pressure of the RIG are determined by the following expressions:

$$\frac{\rho}{n} = \langle\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\rangle \tag{1}$$

$$\frac{p}{n} = \frac{1}{3}\langle\frac{v^2}{c^2}\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\rangle. \tag{2}$$

Here mc^2 is the rest energy of gas particles, and the angle brackets mean averaging over the velocity distribution function (VDF) of QIG particles, n is the number density of particles.

Next, we take into account that, according to the rules of quantum averaging, the number of particles per unit volume, i.e. the number density of particles having momenta in the interval p, p+dp is equal to

$$dn_p = \frac{g}{h^3} f(p) 4\pi p^2 dp. \tag{3}$$

Therefore, the average particle number density of any QRIG is determined by the following integral expression

$$n = \frac{4\pi g}{h^3}\int_0^\infty p^2 f(p) dp. \tag{4}$$

Here f(p) is the invariant DF of QRIG particles, and g is the spin degree of freedom of QIG particles.

Now, passing in expressions (1)-(2) from velocities to momenta of relativistic particles

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{{E_0}^2 + p^2c^2}, \qquad v = \frac{dE}{dp} = c\frac{pc}{E(p)} \tag{5}$$

we obtain that the average energy density and the pressure of the QRIG are determined, respectively, by the following expressions:

$$\rho = \frac{4\pi g}{h^3}\int_0^\infty E(p) f(p) p^2 dp, \tag{6}$$

$$p = \frac{4\pi g}{3h^3}\int_0^\infty \frac{p^2c^2}{E(p)} f(p) p^2 dp. \tag{7}$$

In particular, from expressions (6) and (7) we obtain the following relationship between the average energy density and pressure, i.e. equation of state QRIG:

$$\rho - 3p = \frac{4\pi g}{h^3}(mc^2)^2\int_0^\infty \frac{f(p)}{E(p)} p^2 dp. \tag{8}$$

Hence, we obtain the following fundamental conclusion: the QRIG pressure cannot exceed one third of the average QRIG energy density, i.e.,

$$\rho - 3p \geq 0. \tag{9}$$

Note that the equal sign takes place only for massless QRIGs.

As follows from the above basic position of the invariant relativistic theory of QIG, for the first time we determine the macroscopic characteristics, as well as the equation of state of the QRIG by a purely quantum-statistical method – based on averaging over the DF of the particles of the QIG – the invariance of which was proved in [4-5]. In the proposed theory, neither the method of the maximum entropy principle nor the method of thermodynamic potentials is used anywhere. We emphasize that we find all macroscopic characteristics (n, ρ, P) and the equation of state of the RQIG based on the integrals (4), (6)–(8), considering them to be a function of the chemical potential μ and the absolute temperature T.

The most important distinguishing feature of the proposed new theory is that it is relativistic invariant. Those the absolute temperature and the macroscopic characteristics of the QRIG determined by it - the average particle number density, average energy density, pressure, as well as the QRIG equation of state connecting them - are the same in all inertial frames of reference, as it should be according to the requirement of the invariance of the laws of nature.

However, the existing theories of RQIG are non-invariant, i.e. contradict the principle of relativity for two reasons:
1) temperature is converted according to three rules [1-3];
2) there are "isolated" reference systems [10-12], and this violates the equality of IFR.

Thus, we are talking about a fundamental difference between the proposed invariant relativistic QIG theory and the existing non-invariant relativistic QIG theories, according to which the statistical nature of particles changes upon transition to another IFR. The last statement contradicts the principle of relativity, i.e. invariance of the laws of nature, which cannot be eliminated by any "compromise".

Concluding the section, we present the form of the relativistic invariant DF on momentum of QIG particles

$$f(p) = \left[exp\frac{E(p)-\mu}{kT} - \varepsilon\right]^{-1}. \quad (11)$$

Here E(p) is the energy of the RQIG particle, defined by expression (5), kT is the thermal energy, μ is the chemical potential, ε is the statistics index of the QIG particles, which takes the values 0, 1, -1, respectively, for the distribution of Maxwell-Boltzmann, Bose- Einstein and Fermi-Dirac. Note that the relativistic invariance of the distribution function and absolute temperature was proved in our paper [4] for the case of classical statistics. Obviously, it remains valid for quantum statistics, in which only the method of averaging changes.

2.1 Maxwell-Boltzmann model of a relativistic quantum ideal gas

Let's start studying the properties of the RQIG from the simplest case - from the so-called Maxwell-Boltzmann model of the RQIG. We note right away that, according to quantum mechanics, spin less particles do not exist in nature: particles with integer spins obey Bose statistics, and particles with half-integer spins obey Fermi statistics.

However, this model is useful in that it allows, in many practically important cases that we encounter, to accurately describe the statistical properties of free ones, i.e., not located in an external field, non-interacting structureless systems, i.e. point, relativistic particles, i.e. ideal gases.

As noted above, for "spinless" particles ε=0, i.e. we have the Maxwell-Boltzmann distribution, which has the following form

$$f_{MB} = exp\left[-\frac{E(p)-\mu}{kT}\right]. \quad (12)$$

Further, using the general expressions for macroscopic quantities (4), (6) - (7), as well as the equation of state (8) connecting them, we find by averaging based on the relativistic invariant distribution function of Maxwell-Boltzmann the desired characteristics of the QRIG MB.

Let us start with the determination of the average density of the numbers of RQIG particles n, which is expressed in the case under consideration by the following integral

$$n = \frac{g_0}{h^3} 4\pi e^{\frac{\mu}{kT}} \int_0^\infty p^2 exp\left[-\frac{\sqrt{{E_0}^2+p^2c^2}}{kT}\right] dp. \quad (13)$$

As can be seen from the expression for the energy of a relativistic particle, if we use the relation

$$p = mc\, sh\, x, \quad (14)$$

then the energy can be written as

$$E = mc^2\, ch\, x, \quad (15)$$

which greatly simplify the calculation of the integral (13).

Taking these transformations into account, we obtain

$$n = \frac{g_0}{h^3}(mc)^3 4\pi e^{\frac{\mu}{kT}} \int_0^\infty \mathrm{sh}^2\, x\, ch\, x\, e^{-b\, ch\, x}\, dx, \quad b = \frac{mc^2}{kT}. \quad (16)$$

The integral appearing here is expressed by a modified Bessel function of the 2nd order and is equal to $\frac{k_2(b)}{b}$.

Therefore, the average particle number density of the MB RQIG is determined by the following relation

$$n = \frac{g_0}{h^3}(mc)^3 4\pi \, \frac{k_2(b)}{b} e^{\frac{\mu}{kT}}. \tag{17}$$

Here $g_0 = 1$, since we are considering "spinless" particles, b is the ratio of the rest energy of the particle RQIG and the thermal energy kT.

Note that the found expression for n makes it possible to determine the chemical potential μ for an arbitrary value of the parameter b, which can vary from zero to infinity in subatomic, i.e., intra-nucleon systems, in electron-ion relativistic plasma, as well as in the so-called quark-gluon plasma [20-38]:

$$\mu = kT \ln x, \qquad x = \frac{n}{\frac{g_0}{h^3}4\pi(mc)^3 \frac{k_2(b)}{b}}. \tag{18}$$

It follows that the chemical potential is positive for x>1 and negative for x<1. It takes on a zero value at x=1, i.e. in this case

$$n = 4\pi \, g_0 \left(\frac{mc}{h}\right)^3 \frac{k_2(b)}{b}. \tag{19}$$

Note that the thermodynamic potential method (which we do not use) [11] gives exactly the same result, according to which

$$\mu = \frac{\partial F}{\partial n} \equiv kT \ln x, \tag{20}$$

where F - is the thermodynamic potential, also called free energy.

Proceeding in a similar way, we obtain the following expressions for the average energy density and pressure of the RQIG MB:

$$\rho = 4\pi \, \frac{g_0}{h^3}(mc)^3(mc^2)e^{\frac{\mu}{kT}}\left[\frac{k_1(b)}{b} + 3\frac{k_2(b)}{b^2}\right], \tag{21}$$

$$p = 4\pi \, \frac{g_0}{h^3}(mc)^3(mc^2)e^{\frac{\mu}{kT}}\frac{k_2(b)}{b^2}. \tag{22}$$

To find these results, it suffices to take into account that [24]

$$k_n(b) = \frac{b^n}{(2n-1)!!}\int_0^\infty \mathrm{sh}^{2n} x \, e^{-b \, ch \, x} dx \; =$$

$$= \frac{b^{n-1}}{(2n-3)!!}\int_0^\infty \mathrm{sh}^{2(n-1)} x \, ch \, x \, e^{-b \, ch \, x} dx \, . \tag{23}$$

Here k_n (b) are modified Bessel functions of the nth order (n = 0, 1, 2, …).

Finally, we find the following equation of state based on relation (8):

$$\rho - 3p = 4\pi \frac{g_0}{h^3}(mc)^3(mc^2)e^{\frac{\mu}{kT}}\frac{k_1(b)}{b}. \tag{24}$$

From the obtained exact expressions for the macroscopic characteristics and the equation of state connecting them, the following relations follow:

$$\left.\begin{aligned} \frac{p}{n} &= kT, \\ \frac{\rho}{n} &= mc^2\left[\frac{k_1(b)}{k_2(b)} + \frac{3}{b}\right], \\ \frac{p}{\rho} &= \frac{1}{3+b\frac{k_1(b)}{k_2(b)}}. \end{aligned}\right\} \tag{25}$$

We will need expressions (25) in the future when studying the issue of the maximum velocity of the flow of QRIG particles. Note that these relations exactly coincide with our results obtained in [4], where they were derived on the basis of averaging classical statistics.

First, we study the behavior of the chemical potential and the average density of particles in the essentially nonrelativistic limit, i.e. for $kT \ll mc^2$, as well as in the ultrarelativistic limit, i.e. for $mc^2 \ll kT$ (or for m→0, i.e., in the massless limit). To do this, we take into account the asymptotics of $k_2(b)$ with b≫1 and b≪1 (or b→0):

$$k_2(b) \approx \sqrt{\frac{\pi}{2b}} e^{-b}\left(1 + \frac{15}{8b} + \cdots\right) \quad (b \gg 1)$$

$$k_2(b) \approx \frac{2}{b^2} \quad (b \ll 1) \tag{31}$$

Then using (18), we get for b≫1:

$$\frac{\mu}{kT} \approx \ln\frac{n}{\frac{g_0}{h^3}(mc)^3 4\pi\frac{1}{b}\sqrt{\frac{\pi}{2b}}} + \frac{mc^2}{kT}, \tag{32}$$

or, transforming the expression under the sign of the logarithm, we have

$$\frac{\mu - mc^2}{kT} = \ln\frac{n}{g_0 4\pi\sqrt{\frac{\pi}{2}}\left(\frac{\sqrt{mkT}}{h}\right)^3}. \tag{33}$$

Hence, if the chemical potential μ is equal to the rest energy mc^2, i.e. for $\mu = mc^2$, it follows

$$n = g_0 4\pi \sqrt{\frac{\pi}{2}} \left(\frac{\sqrt{mkT}}{h}\right)^3. \quad (34)$$

In the ultrarelativistic limit, we have

$$\frac{\mu}{kT} = \ln \frac{n}{g_0 4\pi \left(\frac{kT}{hc}\right)^3 \cdot 2}. \quad (35)$$

From the last expression at zero chemical potential, i.e. for μ=0 we have

$$n = g_0 4\pi \left(\frac{kT}{hc}\right)^3 \cdot 2. \quad (36)$$

Thus, the Maxwell-Boltzmann model of the RQIG gives a correct idea of the asymptotic behavior of the chemical potentials and the average particle number density both in the essentially nonrelativistic and in the ultrarelativistic limit [see the following sections of the article].

Properties of RQIG of bose and fermi particles.

Now let's move on to studying the properties of the RQIG of particles that obey relativistic invariant momentum distribution functions - Bose-Einstein and Fermi-Dirac. To do this, we must first determine their macroscopic characteristics, as well as the equations of state connecting them based on quantum averaging over the general distribution (11) at ε=1, -1.

To achieve this goal, we will use the following non-trivial integration method to determine n, ρ and p, namely: we multiply the numerators and denominators of the integrands by $\exp\left[-\frac{E(p)-\mu}{kT}\right]$and consider the fraction $\frac{1}{1-\varepsilon\exp\left[-\frac{E(p)-\mu}{kT}\right]}$ as the sum of an infinitely decreasing geometric progression with denominator $q = \varepsilon \exp\left[-\frac{E(p)-\mu}{kT}\right]$. Obviously, the sum under consideration will be convergent only under the condition |q|<1.

Then, taking into account the integral expressions for macroscopic quantities (4), (6)–(7), we obtain the following general relations:

1) Average particle number density of RQIG
2) $n = \frac{g_{1,-1}}{h^3} 4\pi \int_0^\infty dp\, p^2 e^{-\frac{E(p)-\mu}{kT}} \sum_{n=0}^{\infty} e^{-n\frac{E(p)-\mu}{kT}} q^n =$

$$= \frac{g_{1,-1}}{h^3} 4\pi \sum_{n=0}^{\infty} \int_0^\infty dp\, p^2 \varepsilon^n \exp\left[-(n+1)\frac{E(p)-\mu}{kT}\right]. \quad (37)$$

Here $g_{1,-1}$ -is the number of spin degrees of freedom for Bose and Fermi particles; ε=1, -1 - respectively for the distribution of BE and FD.

Further, using transformations (14)-(15), as well as the result (23), we obtain the following general integral relation for determining the average density of the numbers of particles of the RQIG of Bose and Fermi particles.

$$n = 4\pi g_{1,-1} \left(\frac{mc}{h}\right)^3 \sum_{n=0}^{\infty} \varepsilon^n \frac{k_2[(n+1)b]}{(n+1)b} e^{(n+1)a}. \quad (38)$$

Here: $a = \frac{\mu}{kT}$, $b = \frac{mc^2}{kT}$.

2) Proceeding in a completely similar way and using the general expressions for the macroscopic characteristics of the RQIG (6)-(7), taking into account the sum (37), we obtain:

$$\rho = 4\pi g_{1,-1} \left(\frac{mc}{h}\right)^3 (mc^2) \sum_{n=0}^{\infty} \varepsilon^n e^{(n+1)a} \left\{\frac{k_1[(n+1)b]}{(n+1)b} + 3\frac{k_2[(n+1)b]}{(n+1)^2 b^2}\right\}, \quad (39)$$

$$p = 4\pi g_{1,-1} \left(\frac{mc}{h}\right)^3 (mc^2) \sum_{n=0}^{\infty} \varepsilon^n e^{(n+1)a} \frac{k_2[(n+1)b]}{(n+1)^2 b^2}. \quad (40)$$

3) And finally, from the expressions for the average energy density ρ and pressure p, we find the following equation of state for the RQIG of Bose and Fermi particles, which relates them, based on (8):

$$\rho - 3p = 4\pi g_{1,-1} \left(\frac{mc}{h}\right)^3 (mc^2) \sum_{n=0}^{\infty} \varepsilon^n e^{(n+1)a} \frac{k_1[(n+1)b]}{(n+1)b}. \quad (41)$$

Thus, we managed to obtain general expressions for determining the macroscopic characteristics (n, ρ, p), as well as the equation of state (41), which are valid for any values of the parameters a and b, naturally satisfying the summation condition of the above series.

Now, following the logic of things, we study the properties of the RQIG of Bose and Fermi particles separately.

2.2 "Special" relativistic quantum ideal gas of Bose particles with integer spins

The "special" behavior of QIG Bose particles is well known, and in particular, their surprising property is the accumulation of particles, i.e. condensation, not only at zero momentum, but also with a finite momentum value, continues to be one of the topical problems of modern physics: the Bose-Einstein condensation of the RQIG, the condensation of rotons in a superfluid liquid [25-27].

However, the experimental results obtained recently - on the multiple production of particles by ultrahigh energies in the CERN Large Hadron Collider, the detection of electron-positron relativistic cosmic ray plasma, controlled inertial thermonuclear fusion [29, 30], [35-38], were put on the agenda day a number of questions awaiting their decision. One of these issues is, in our opinion, the study of the properties of RQIG in systems where the ratio of thermal energy and rest energy of a gas particle can vary from zero to infinity (these are the so-called subnucleon systems), and the chemical potential can take on the value, figuratively speaking, from minus to plus infinity. In such cases, of course, it is required to obtain expressions for the macroscopic characteristics of the RQIG for arbitrary values of the parameters $a = \mu/kT$, $b = mc^2/kT$. Here μ is the chemical potential, mc^2 - is the rest energy of a gas particle, kT - is the thermal energy.

Further, we take into account that the Bose-Einstein distribution describes the quantum statistics of particles with integer spins, which corresponds to ε=1 in (11), i.e.

$$f_{BE} = \frac{1}{exp\left[\frac{E(p)-\mu}{kT}\right]-1}. \quad (42)$$

Hence it follows that the number of particles is not negative, at any momentum, even at $p = 0$, the chemical potential must satisfy the condition $\mu < mc^2$. In addition, when $E(p) = \mu$, a "singularity" appears in the Bose-Einstein distribution, i.e. the denominator of expression (42) vanishes. Therefore, the case $E(p) \to \mu$ requires its investigation with allowance for the ESE asymptotics near this "critical" point.

Thus, while the condition $\mu < mc^2$ is assumed to be satisfied, from (38) we obtain:

$$n = 4\pi g_1 \left(\frac{mc}{h}\right)^3 \sum_{n=0}^{\infty} \frac{k_2[(n+1)b]}{(n+1)b} e^{(n+1)a}. \quad (43)$$

It follows from the last expression that the average number density of particles in the RQIG BE is determined by the absolute temperature T and the chemical potential μ.

Let us first consider the essentially nonrelativistic limit, which, as shown above, corresponds to very large values of the parameter b, i.e. b>>1 (or b→∞)

$$kT \ll mc^2. \quad (44)$$

Further, recalling the asymptotics of the modified 2nd order Bessel function (31) from (43), we obtain the following expression:

$$n = 4\pi g_1 \left(\frac{mc}{h}\right)^3 \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{2}} \frac{e^{(n+1)(a-b)}}{\sqrt{(n+1)b}(n+1)b}. \quad (45)$$

It follows from this that if the condition a=b is satisfied, i.e.

$$\mu(T_c) = mc^2, \quad (46)$$

then we get

$$n = 4\pi g_1 \left(\frac{mc}{h}\right)^3 \sqrt{\frac{\pi}{2}} \frac{1}{b^{3/2}} \sum_{n=0}^{\infty} \frac{1}{(n+1)^{3/2}}. \quad (47)$$

Further, taking into account that the last sum is equal to $\xi\left(\frac{3}{2}\right)$ (here $\xi(x)$ - is the Riemann zeta function) and transforming (47), i.e., setting $b = \frac{mc^2}{kT}$, we have the following condition for the critical particle number density of the RQIG BE:

$$n_c = \left(\frac{\sqrt{mkT_c}}{h}\right)^3 4\pi g_1 \sqrt{\frac{\pi}{2}} \xi\left(\frac{3}{2}\right). \quad (48)$$

Now, returning to the remark made at the beginning, when $E(p) \to \mu = mc^2$, we see that this corresponds to the asymptotic behavior of ultrasmall particle momenta in the BE distribution. In other words, it is the accumulation of particles in the macrostate with zero momentum, and $n_c$represents the number of condensed particles with zero momentum .

Therefore, we will come to the conclusion that the condition for particle condensation is $\mu(T_c) = mc^2$, and not μ=0, as it should be. We will verify below that the zero chemical potential corresponds to the ultrarelativistic limit, i.e. μ≪kT.

Indeed, for very small values of the parameter b, i.e. for b≪1 (or b→0), taking into account the second part of expression (31), as well as (43), we obtain ($kT \gg mc^2$ or m→0!):

$$n = 4\pi g_1 \left(\frac{mc}{h}\right)^3 2\sum_{n=0}^{\infty} \frac{1}{(n+1)b^3} e^{(n+1)a}. \qquad (49)$$

It follows that the condition $E(p) \to \mu = mc^2$ in the massless limit means zero chemical potential, i.e. μ=0. Then from (49) we have

$$n_{BE} = 4\pi g_1 2\xi(3)\left(\frac{kT}{hc}\right)^3. \qquad (50)$$

The latter corresponds to the total number of bosons with zero mass, since in this case the momentum value p=0 is assumed by the particles with zero probability. Note that the last result differs significantly from n_c BEC, since the latter corresponds to condensation at ultralow temperatures. We emphasize that the expression for the particle number density of ultrarelativistic Bose gases (50) was previously obtained on the basis of the Gibbs canonical distribution [25] and is consistent with our result (see also [26, 27, 31, 34]).

And the general case, when the ratios of the chemical potential and rest energy to thermal energy are finite, i.e. intermediate values, and the chemical potential is less than (or within the limit equal to) the rest energy, as far as we know, has not been considered in the literature.

As for the case $\mu > mc^2$, which is possible only for a Fermi gas, the integrals involved in finding the macroscopic characteristics of the QRIG FD diverge. As is well known, they will be finite in the limit of absolute temperatures tending to zero (see, for example, Molinari [28]), which are of no interest in the context of the problem under consideration.

To obtain the Einstein result, we write the exponent in the BE distribution in the following identical form, following Molinari [28],

$$\frac{E-\mu}{kT} \equiv \frac{(E-mc^2)-(\mu-mc^2)}{kT}.$$

Then the 1st term will mean the kinetic energy of the relativistic particle, and the 2nd term expresses the "shifted" chemical potential by the rest energy of the particle, i.e. $\mu' = \mu - mc^2$. If we now consider the essentially nonrelativistic limit ($kT \ll mc^2$ and $p \ll mc$), which adequately describes the BEC at ultralow temperatures tending to zero, the result (48) first obtained by Einstein follows exactly.

Therefore, the last results are consequences of the condition $E(p) \to \mu = mc^2$, respectively in the essentially nonrelativistic and ultrarelativistic limit.

Thus, the conclusion: "for a Bose gas, the chemical potential must be $\mu \leq 0$ at an arbitrary temperature" [25] is incorrect from a physical point of view. In fact, at any temperature it is necessary to satisfy the condition

$$\mu \leq mc^2, \qquad (51)$$

which ensures non-negativity and limitation of the average number density of particles of the RQIG BE. In addition, as expressions (48) and (51) show, at temperatures tending to absolute zero, the average number of particles with nonzero momentum tends to zero. In other words, at T→0 all Bose particles accumulate in a state with zero momentum, i.e. form the so-called pure condensate, and there are no above condensate Bose particles.

Using the expressions found above for macroscopic characteristics (n, ρ, p), as well as the equation of state connecting them, we can find them for arbitrary values of the parameters a $\leq b$, since according to (51) $\mu \leq mc^2$. For example, in works [27], [31], these parameters changed in the following intervals: $0{,}1 \leq a \leq 1$, $0{,}1 \leq b \leq 10$. Naturally, in numerical simulation, the study of the case $b \gg 1$ and $b \ll 1$is obviously difficult. This is due to the fact that in limiting cases, i.e. for $b \gg 1$ and $b \ll 1$ (or b→0), it is necessary to sum up the entire series appearing in the expressions for the macroscopic characteristics of the Bose gas. The latter seems to be irrational from the point of view of numerical calculation. In such "subtle" cases, it is reasonable to find them analytically, as demonstrated above (see also below). However, we restrict ourselves to the study of the case $\mu \to 0$, i.e. behavior of a massless ultrarelativistic Bose gas at arbitrary temperatures.

Since in the limit b→0, p=ρ/3, it suffices to find one of them. We then find the average energy density of the QRIG BE in the ultrarelativistic limit. To do this, we use expression (39) and take into account that for very small values of the parameter b, i.e. at b≪1 (or b→0)

$$k_1(b) \approx \frac{1}{b}; \qquad k_2(b) \approx \frac{2}{b^2}. \qquad (52)$$

Further, assuming $\mu = mc^2 \to 0$, we get:

$$\rho = 4\pi g_1 \left(\frac{mc}{h}\right)^3 (mc^2)\left\{\frac{1}{b^2}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{6}{b^4}\sum_{k=1}^{\infty}\frac{1}{k^4}\right\}. \qquad (53)$$

But as known [24]:

$$\sum_{k=1}^{\infty}\frac{1}{k^s} = \xi(s). \qquad (54)$$

Thus, passing to the limit b→0, we find

$$\rho_{BE} = 24\pi g_1 \left(\frac{kT}{hc}\right)^3 (kT)\xi(4). \qquad (55)$$

Therefore, the average internal energy of one ultrarelativistic Bose particle is

$$u_{\mathrm{BE}} = \frac{\rho}{n} = \frac{3\xi(4)}{\xi(3)}kT. \qquad (56)$$

At the same time, for the ultrarelativistic QIG MB, according to (25)

$$u_{\mathrm{MB}} = \frac{\rho}{n} = 3kT. \qquad (57)$$

2.3 Relativistic quantum ideal gas of fermi particles with half-integer spins

Now let's consider some properties of the RQIG whose particles obey the Fermi-Dirac distribution, which has the following form

$$f_{BE} = \frac{1}{exp\left[\frac{E(p)-\mu}{kT}\right]+1}. \qquad (58)$$

Further, for definiteness, we restrict ourselves to the study of macroscopic characteristics and the equation of state of the RQIG of Fermi particles at temperatures not equal to absolute zero. In addition, we believe that, as in the case of the RQIG BE, the inequality

$$\mu \le mc^2. \qquad (59)$$

Then in the essentially nonrelativistic limit, i.e. when $kT \ll mc^2$ for the average particle number density of the RQIG FD we obtain the following expression

$$n = 4\pi g_{-1}\left(\frac{mc}{h}\right)^3 \frac{1}{b}\sqrt{\frac{\pi}{2b}}\left(1-\frac{1}{\sqrt{2}}\right)\xi\left(\frac{3}{2}\right), \qquad (60)$$

where $b = \frac{mc^2}{kT} \gg 1$.

In the ultrarelativistic limit, i.e. $b \ll 1$ (or m→0) we get

$$n_{FD} = 2\pi g_{-1}\left(\frac{kT}{hc}\right)^3 \frac{3}{2}\xi(3). \qquad (61)$$

Comparing the latest results obtained for the average particle number density of the RQIG FD with the corresponding expressions for the RQIG BE (48), (50) and the RQIG MB (34), (36), we come to the conclusion that within these limits the properties of these gases are similar to each other friend.

Concluding the section, we give an expression for the average energy density of the RQIG FD in the ultrarelativistic or massless limit

$$\rho_{FD} = 4\pi g_{-1}\left(\frac{kT}{hc}\right)^3 (kT)\frac{3}{4}7\xi(4). \qquad (62)$$

Therefore, the average internal energy of one ultrarelativistic fermi particle is equal to

$$u_{FD} = \frac{\rho}{n} = kT\frac{7}{2}\frac{\xi(4)}{\xi(3)}. \qquad (63)$$

Thus, within the framework of the proposed invariant relativistic theory of a quantum ideal gas, for the first time, a closed description of their properties was given based on quantum statistical averaging without using the method of thermodynamic potentials (compare with [10, 11, 12, 15, 17, 26, 27, 31, 39]).

For the sake of justice, it should be noted the works [28, 34], in which the macroscopic characteristics of quantum relativistic bosons and fermions were also found on the basis of averaging over the FD and BE energy distributions. However, the conditions for the condensation of Bose gases and the ultrarelativistic limits for quantum Bose and Fermi particles were not considered in them. In short, the studies were limited to the analysis of the behavior of the macro-characteristics of quantum relativistic bosons and fermions in the essentially non-relativistic limit, i.e. $kT \ll mc^2$ (or $b \gg 1$), when the condition $\mu' = \mu - mc^2 < 0$ is satisfied for both cases (which means weak degeneracy for fermions).

All results are obtained by a unified approach - using only the properties of the modified Bessel functions of integer order k_n (b), which greatly simplifies the calculation of the desired characteristics of the RQIG and is very convenient when applied to the study of the set of RQIG in physical systems, where the ratio of the chemical potential and rest energy to thermal energy can change from zero to infinity (see, for example, [20, 21, 22, 23, 29, 30]).

2.4 Relativistic quantum gas model of the Universe

Analyzing the works devoted to the study of the evolution of the Universe in the framework of the so-called standard cosmological models, as well as the properties of the quark-gluon plasma, it is easy to see that the gas model is widely used in them [20, 21, 22, 23, 31]. Indeed, in a number of cases it makes it possible to adequately describe the phenomena under study in a simple way. In other words, this is its main charm.

The relativistic quantum gas model of the Universe offered by us, represents some generalization of the existing models of the Universe. In it, it is considered as a closed system of a set of relativistic quantum ideal gases, particles, antiparticles, and photons included in it.

We note that the inclusion of antiparticles in the relativistic theory of QIG, as shown in a number of recent papers [26, 27], makes it possible to study the processes of particle production and annihilation, as well as the phenomenon of particle condensation in physically important situations.

To determine the macroscopic characteristics of the RQIG of Bose and fermi -antiparticles, we use the fact that the correct relationship between the statistics and the type of particles is preserved, i.e. a particle and its antiparticle obey the same statistics. In addition, it is enough to change the chemical potential for antiparticles to $-\mu$, i.e. $\bar{\mu} = -\mu$ (hereinafter, for antiparticles, we use the generally accepted notation - the overline). [31]

Let's start with a description of the properties of the RQIG of Bose particles and antiparticles, which obey the Bose-Einstein distribution

$$f_{BE} = \frac{1}{e^{\frac{E-\mu}{kT}}-1}, \quad \bar{f}_{BE} = \frac{1}{e^{\frac{E+\bar{\mu}}{kT}}-1}. \tag{64}$$

Since the number of particles and antiparticles cannot be negative, the following inequalities must hold for particles and antiparticles, respectively:

$$\mu < mc^2, \quad \bar{\mu} > -mc^2. \tag{65}$$

Combining these inequalities, introducing a single notation for the chemical potential of the RQIG of Bose particles and antiparticles, we obtain the condition

$$-mc^2 < \mu < mc^2. \tag{66}$$

The case $|\mu| = mc^2$, i.e. when the denominator of the distribution for f and $\bar{f}$ vanishes, naturally, as was shown above, it requires special consideration with the help of limits (see the previous sections).

This "special" case, as we know, corresponds to the conditions of condensation, respectively: for Bose particles ($\mu = mc^2$) and for Bose antiparticles ($\bar{\mu} = -mc^2$), i.e. their accumulation in the state with zero momentum. From here we get the fundamental conclusion that the condensation of Bose particles and antiparticles cannot occur simultaneously, because they correspond to different chemical potentials, the difference of which in this case is equal to $\mu - \bar{\mu} = 2mc^2$, i.e. twice the rest energy of a free particle. Simply put, they can coexist together only if the general inequality (66) is satisfied. Those Bose particles cannot exist at chemical potential values $\mu > mc^2$, and Bose antiparticles cannot exist at $\bar{\mu} < -mc^2$,, since under these conditions their numbers will be negative, which is impossible from a physical point of view.

As we can see, the inclusion of antiparticles in the RQIG theory significantly changes the range of possible values of the chemical potential μ, which is limited by inequality (66).

Further, we must take into account that in the possible processes of creation and annihilation of a particle-antiparticle pair, the following fundamental laws must be fulfilled:

1. Particle and antiparticle are always born in pairs.

2. During the annihilation of a pair of particles and antiparticles, they turn into two photons, i.e. into an electrically neutral particle with zero rest mass.

Therefore, in these processes, the law of conservation of electric charge must be fulfilled, which in this case ensures the electrical neutrality of the system of pairs of particles and antiparticles.

To eliminate misunderstandings, we note that we consider the processes of creation and annihilation of only a particle-antiparticle pair without the participation of other particles.

Then, using our previous results obtained for the RQIG BE, we obtain the following expressions for the average number density of particles and antiparticles:

$$\left.\begin{aligned} n &= 4\pi g_1 \left(\frac{mc}{h}\right)^3 \sum_{n=0}^{\infty} \frac{k_2[(n+1)b]}{(n+1)b} e^{(n+1)a}, \\ \bar{n} &= 4\pi g_1 \left(\frac{mc}{h}\right)^3 \sum_{n=0}^{\infty} \frac{k_2[(n+1)b]}{(n+1)b} e^{(n+1)\bar{a}}. \end{aligned}\right\} \tag{67}$$

Here:

$$b = \frac{mc^2}{kT}, \qquad a = \frac{\mu}{kT}, \bar{a} = \frac{\bar{\mu}}{kT}. \tag{68}$$

Since a particle and an antiparticle have the same rest mass and opposite electric charges, to ensure the electrical neutrality of a system of pairs of particles and antiparticles, the following condition must be met:

$$Q = nq + \bar{n}\bar{q} \equiv q(n - \bar{n}) = 0. \tag{69}$$

Therefore, taking into account that $\bar{\mu} = -\mu$, we obtain:

$$n - \bar{n} = 4\pi g_1 \left(\frac{mc}{h}\right)^3 \sum_{n=0}^{\infty} \frac{k_2[(n+1)b]}{(n+1)b} \{e^{(n+1)\mu} - e^{-(n+1)\mu}\}. \tag{70}$$

Thus, the electrical neutrality of a system of pairs of particles-antiparticles is preserved only and only if the chemical potential of the system

$$\mu = 0, \tag{71}$$

at any temperature.

Naturally, such a conclusion can be reached in another way, namely: under conditions of statistical equilibrium, the temperature and chemical potential for the entire system (in this case, the Universe) must be the same. Since for photons (as for a particle with zero rest mass) $\mu = 0$, then for all particles and antiparticles the chemical potential is equal to zero.

Note that a similar result also follows from the theory of thermodynamic potentials, according to which, under conditions of thermodynamic equilibrium, the free energy potential should have a minimum value, i.e.

$$\frac{\partial F}{\partial n} = 0, \tag{72}$$

for any gas in the system under consideration. However, according to (20), this derivative is equal to the chemical potential and is necessarily equal to zero - otherwise the system of ideal gases will not be in thermodynamic equilibrium.

Zero chemical potential significantly changes the role of massive, i.e. having mass, particles and massless, i.e. particles with zero rest mass, namely: particles with rest mass $mc^2 \gg kT$ make an exponentially small contribution compared to massless particles $m \to 0$ in all macroscopic characteristics.

For example, for the average number density of bose- particles, we have

$$\left.\begin{aligned} n_{BE} &= 4\pi g_1 \left(\frac{kT}{hc}\right)^3 \sqrt{\frac{\pi}{2}} b^{-\frac{3}{2}} e^{-b} \quad (b \gg 1), \\ n_{BE} &= 4\pi g_1 \left(\frac{kT}{hc}\right)^3 2\xi(3). \quad (b \ll 1) \end{aligned}\right\} \tag{73}$$

This behavior of the number of particles in the literature is called the exponential "suppression" of the population of superheavy particles.

Similarly, for the average energy density of fermi-particles, we obtain

$$\left.\begin{aligned} \rho_{FD} &= 4\pi g_{-1} \left(\frac{kT}{hc}\right)^3 (kT) \sqrt{\frac{\pi}{2}} b^{-\frac{5}{2}} e^{-b} \quad (b \gg 1), \\ \rho_{FD} &= 24\pi g_{-1} \left(\frac{kT}{hc}\right)^3 (kT) \frac{7}{8} \xi(4) \quad (b \ll 1). \end{aligned}\right\} \tag{74}$$

Consequently, in systems with zero chemical potential [21], the predominant role is played by massless ultrarelativistic particles.

And finally, as the temperature tends to absolute zero, we come to the conclusion that all particles form the so-called pure condensate, and there are no above condensate particles. The author of [20] figuratively calls this situation the “freezing” of the Universe or quark-gluon plasma, as well as the “sea” of Fermi.

Concluding the section, we emphasize that according to the theories of the "grand" unification, the chemical potentials for all particles and antiparticles are equal to zero [21]. Moreover, at ultrahigh temperatures, i.e. at $kT \gg mc^2$, all particles with a good approximation can be considered as ultrarelativistic, i.e. their energy should be written as $E \approx pc$. In other words, under these conditions, we can set $\mu \to 0$ and $m \to 0$. Obviously, for massless bosons and fermions (equalities $\mu=0$ and $m=0$) these results will be exact.

2.5 Limiting velocity of the flow of QRIG particles with an adiabatic change in its state.

As is known, one of the most important problems of relativistic high-energy physics, as well as laser-controlled relativistic thermonuclear fusion, is the so-called energization (i.e., thermal and field acceleration) of particles [35, 37, 38, 40]. So far, all aspects of this really topical scientific and practical problem have not been fully resolved. This is due to the fact that the efficiency of thermonuclear reactions and the so-called "luminosity" of the processes of multiple production and annihilation of particles in collisions of ultrarelativistic particles (for example, heavy particles in the Large Hadron Collider [37, 38] and ultrafast thermal neutrons [AONSA, neutron school , 21

November, 2022]) are ultimately determined by the number of particles produced and the effective collision cross section of high-energy colliding particles.

In particular, in laser-controlled inertial thermonuclear fusion, an ultrafast compression of the resulting plasma by a system of lasers occurs under the action of super-powerful pulses (the power of which is on the order of terawatts, and the duration of the pulses is several picoseconds). At the same time, the processes of heating the laser plasma (up to hundreds of millions of kelvins) and the ultrafast acceleration of the formed flows of plasma particles take place simultaneously. As a result, they enter into nuclear reactions in one trillionth of a second, i.e. until the reverse "expansion" of the plasma.

According to existing theoretical models, an adiabatic change in the state of the resulting plasma of ionized particles occurs in the system under consideration. When studying such processes, a sufficiently good and adequate approximation is the gas model of the plasma of relativistic particles [32, 33, 35, 36].

Since, in this case, the flow rates of particles of quantum ideal gases are very close to the speed of light, i.e. $v \approx c$, their pressure is comparable to their energy density (see expressions (1)-(2)).

Therefore, in such situations, it is necessary to use the equation of the adiabatic process for a quantum relativistic ideal gas (Bose and Fermi particles), as well as take into account the relativistic Bernoulli equation (binding pressure, average energy density, average particle number density and particle flow velocity QRIG). Note that a similar problem also arises in the phenomenon of thermal accretion, i.e. accumulation of particles in a state with ultrahigh pressure [36]. A similar question arises in the problem of the "expansion" of the Universe into the void[40].

Following the foregoing, we represent the set of QRIG as a closed adiabatic system. As is known, with an adiabatic change in the state of a gas [31]:

$$dQ = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) = 0. \tag{75}$$

From here we get the following equation:

$$\frac{d\rho}{dn} = \frac{p+\rho}{n}, \tag{76}$$

which makes it possible to determine the average energy density of particles, as well as the pressure of the RQIG in the adiabatic process, and as a result of its integration, we find the following equation of state for the RQIG in the case under consideration

$$\rho = nmc^2 + \frac{p}{\gamma-1}, \qquad p = \alpha n^{\gamma}, \quad \alpha = const. \tag{77}$$

Here: $\gamma -$ is the adiabatic index of the RQIG, which is determined by the following expression

$$\gamma = 1 + \frac{k}{c_V}, \qquad c_V = \frac{du}{dT}, \qquad u = \frac{\rho}{n}. \tag{78}$$

Now we take into account that the velocity of the flow of QRIG particles obeys the relativistic Bernoulli equation [31]

$$\frac{1}{\sqrt{1-u^2}} = const \cdot \frac{n}{p+\rho}, \quad u = \frac{v}{c}. \tag{79}$$

Here: ϑ – is the speed of the flow of QRIG particles, c is the speed of light.

The constant in this expression is determined from the condition that the initial flow rate of the QRIG is equal to zero. Then

$$const = \frac{p_0+\rho_0}{n_0}.$$

The upper limit $\frac{n}{p+\rho}$ is reached as $p \to 0$ (and hence $\rho \to n \cdot mc^2$), i.e. while flying into the void. Thus, the maximum flow rate of QRIG particles is determined by the following expression

$$(1-{u_m}^2)^{-\frac{1}{2}} = \frac{p_0+\rho_0}{n_0 \cdot mc^2}. \tag{80}$$

Further, v_{max} will be expressed in terms of the speed of sound v_s in a gas with parameters p_0, ρ_0, n_0. To do this, we use its definition [31]

$$a^2 = \frac{{v_s}^2}{c^2} = \frac{dp}{d\rho} \equiv \frac{\frac{dp}{dn}}{\frac{d\rho}{dn}} = \frac{\gamma p}{n \cdot mc^2 + \frac{\gamma p}{\gamma-1}}. \tag{81}$$

Now, taking into account the equation of state in the adiabatic process (77), we write expression (80) in the following form

$$(1-{u_m}^2)^{-\frac{1}{2}} = \frac{1}{n \cdot mc^2}\left(n \cdot mc^2 + \frac{\gamma p}{\gamma-1}\right) \equiv \frac{\gamma p}{nmc^2a^2}. \tag{82}$$

Finally, denoting $\gamma p = x$, we find from (81):

$$\gamma p = nmc^2 \frac{\gamma-1}{\gamma-(1+a^2)} a^2. \tag{83}$$

Then from expression (82) we obtain

$$(1-u_m{}^2)^{-\frac{1}{2}} = \frac{\gamma p}{a^2 nmc^2} \equiv \left(1-\frac{a^2}{\gamma-1}\right)^{-1}. \qquad (84)$$

Here we have changed p_0, ρ_0, n_0 to p, ρ, n since they are arbitrary.

Hence, rewriting x in the identical form

$$x = nmc^2 \frac{a^2}{1-\frac{a^2}{\gamma-1}}, \qquad (85)$$

taking into account (85) and substituting into (84) we obtain

$$\left.\begin{aligned} \gamma \frac{p}{nmc^2} &= \frac{a^2}{1-\frac{a^2}{\gamma-1}} \quad (A), \\ \sqrt{1-u_m{}^2} &= 1-\frac{a^2}{\gamma-1} \quad (B). \end{aligned}\right\} \qquad (86)$$

The last two relations make it possible to determine the flow rate of RQIG particles, as well as the ratios of pressure and the average density of the numbers of RQIG particles, which are necessary in the study of accretion phenomena, i.e. accumulation of particles in a state with superhigh pressure and energy density [32, 35], as well as in the problem of the "expansion" of the Universe into the void [40].

From the analysis of system (86) it follows that the condition of the impossibility of motion at the speed of light and the condition of positive pressure require the fulfillment of the inequality

$$\gamma > 1 + a^2. \qquad (87)$$

But according to the equation of state RQIG (41), the limit equation of state has the form

$$p = \frac{\rho}{3}, \qquad (88)$$

which corresponds to the massless limit (i.e. m→0, T≠0) or ultrarelativistic limit (i.e. $kT \gg mc^2$).

Consequently, from (81) it follows that in this case

$$a^2 = \frac{1}{3}. \qquad (89)$$

Thus, we finally obtain the following fundamental inequality

$$\gamma > \frac{4}{3}, \qquad (90)$$

under which, relations (A) and (B) of system (86) will be consistent, i.e. the pressure will be positive, and the particle flow speed will be up to the speed of light. In other words, there can't be RQIGs with adiabatic exponent $\gamma \leq \frac{4}{3}$.

Similarly, investigating the dynamic equilibrium condition of a star, we obtain the following inequality $\frac{R}{R_g} \geq \gamma_{ph} \left(where\ R_g = \frac{2GM}{c^2}\right), R_g -$ gravity radius, under which there is a stable equilibrium position of the star (here: G is the gravitational constant, c -is the speed of light, $\gamma_{ph} -$ is the smallest adiabatic exponent QRIG, i.e. photons [see below]).

Next, we find the adiabatic exponents, respectively, for the Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein RQIG in this limit, using expressions (25), (63) and (56) based on (78) [5]:

$$\gamma_{MB} = \frac{4}{3}, \qquad \gamma_{FD} = 1 + \frac{2}{7}\frac{\xi(3)}{\xi(4)}, \qquad \gamma_{BE} = 1 + \frac{1}{3}\frac{\xi(3)}{\xi(4)}. \qquad (91)$$

Now, substituting the found values of the adiabatic exponents into the system (86), we come to the following conclusions:

1. For ultrarelativistic QIG MB $\gamma = \frac{4}{3}$, i.e. $\frac{a^2}{\gamma-1} = 1$. In this case, p→∞ and u→1.

But there are no spinless particles in nature, and this example is used here as an illustration.

2. For massless QIG FD $\gamma < \frac{4}{3}$, i.e. $\frac{a^2}{\gamma-1} > 1$. However, for these values of γ, system (86) turns into equalities of meaningless physical content, namely: according to (A), the pressure becomes negative, which is not satisfied by any ideal gas known to date (of course, except for the hypothetical "dark matter"); relation (B) is not satisfied at all, because the left side is non-negative and the right side is negative. In other words, the flow velocity of QIG particles becomes imaginary, which is unacceptable.

The authors of [11] interpret this emerging feature as properties of RIG tachyons (hypothetical superluminal particles), which create the much desired negative pressure necessary for the formation of "dark" matter.

However, we are guided by the fact that we are dealing with real-life particles and substances (the opposite assumption is "blasphemy" even from a scientific point of view).

But the result (63) and the adiabatic exponent γ_{FD} following from it are exact. The only incorrect assumption in the chain of reasoning made here is the unsubstantiated assumption about the existence of Fermi particles with zero rest mass [21], which is made in the framework of the "grand" unification theories.

But as the above proof proves, this assumption is incorrect from a physical point of view, which leads to results that do not correspond to common sense.

3. For a massless Bose particle, i.e. photon $\gamma > \frac{4}{3}$. Therefore, in this case $\frac{a^2}{\gamma - 1} < 1$.

Thus, we will come to the conclusion that the particle flow velocity of a massless photon gas has a limiting velocity less than the speed of light, which is equal to:

$$v_{ph} = c\sqrt{1 - \left[1 - \frac{\xi(4)}{\xi(3)}\right]^2}. \tag{92}$$

Further, taking into account the values of the Riemann zeta functions $\xi(4) = 1{,}0823\ldots$, $\xi(3) = 1{,}2020\ldots$ we obtain

$$u_{ph} = \frac{v_{ph}}{c} \approx 0{,}436 < a_{max} = \frac{v_{smax}}{c} = \frac{1}{\sqrt{3}} \approx 0{,}578\ldots \tag{93}$$

Therefore, the particle flow velocity of even photon gas is subsonic! Thus, we will come to the conclusion that the following fundamental inequalities $\gamma \geq \gamma_{ph} > \frac{4}{3}$ и $v \leq v_{ph} < v_{smax}$ will hold for all real-life QRIGs.

The contradiction that has arisen above can be solved as follows: if a Fermi particle (for example, a relic neutrino, which is the only candidate of Fermi particles that “claims” to have zero rest mass) has a very small but nonzero mass, then now for it $a^2 < \frac{1}{3}$ – since $a^2 = \frac{1}{3}$ only for massless particles. Similarly, $\gamma_{\text{rel.neutr.}} > \gamma_{ph} > \frac{4}{3}$ i.e. $\gamma_{\text{rel.neutr.}} - 1 > \frac{1}{3}$.

Consequently, in this case, at any temperatures other than zero, i.e. T≠0 and T≠∞, according to what has been proven, all QRIG will create non-negative pressure and their particle flow rates will satisfy the condition $v \leq v_{ph} < v_{smax}$.

Thus, Fermi particles with zero rest mass cannot exist in nature. In other words, the only particle with zero rest mass is the Bose particle, the photon.

Thus, we will come to the conclusion that the ratio of pressure to the maximum possible value of the average energy density refers to a massless bosonic gas, i.e. photon gas.

Combining the last and previous conclusions, we come to the conclusion that there cannot be: "collapse" with infinite pressure and average energy density - which would take place, according to the above proved for $\gamma \leq \frac{4}{3}$; and irreversible "expansion" of the Universe after the "Big Bang" (since the speed of the flow of QRIG particles is less than the limiting speed of sound in nature, i.e. the criterion of "expansion" $\frac{v}{v_{smax}} > 1$ is not met).

Concluding the section, we emphasize that the steady-state motion of QRIG particle flows analyzed by us in the adiabatic process of changing its state from a mathematical point of view is equivalent to finding a stationary solution of the equations describing the laws of conservation of energy and particle flow, taking into account the first law of relativistic thermodynamics (for details of analysis and calculations, see [40]). In particular, an important conclusion was obtained in it that when the Universe "expansion" even from a singular point with infinite pressure and energy density, the speed of its "expansion" does not exceed the speed of light. The value found in this work, the so-called "Hubble constant", equal to $71\frac{\frac{km}{s}}{Mega.p.s}$ at $v = c$, in the Big Bang theory is considered as an increase in the "acceleration velocity" for each Mega.p.s distances.

For comparison, we note that, according to the data given in W. Freedman e.a. [40], the most probable values of the Hubble velocity (in units of $\frac{\frac{km}{s}}{Mega.p.s}$) are: 67.4±0.5 (Planck collaboration), 69.8±1.9 (CCHP), 73.9±1.6 (cepheids).

However, this speed is not the speed of the "retreat" of the Universe - which is obvious according to the above. In other words, the maximum particle flow velocity $v_{max} = c$ of a QIG that does not exist in nature is served under the “sauce” of the “constant Hubble”.

This is due to the fact that the ratios $= \frac{\rho}{3}$, $a^2 = \frac{1}{3}$, $\gamma = \frac{4}{3}$, used in the Big Bang theory, which are not satisfied by any of the QRIG existing in reality, since for them $\gamma \geq \gamma_{ph} > \frac{4}{3}$.

Naturally, the fundamental inequalities proved above $\gamma \geq \gamma_{ph} > \frac{4}{3}$;

$v \leq v_{ph} < v_{s\,max}$ could not be obtained without analyzing the behavior of the QRIG in the massless limit.

According to the proposed relativistic quantum model of the Universe, at absolute zero temperature it forms a kind of condensate of its constituent particles or, in other words, vacuum.

If we assume that at a certain critical temperature, due to quantum and thermal fluctuations, there is a continuous birth of a pair of electron neutrinos and antineutrinos from vacuum and their annihilation into two photons (or vice versa - these statements are equivalent) - then, after the establishment of thermal equilibrium, a quantum IG of the electron neutrino is formed - antineutrinos and photons.

Next, consider this collection of Fermi and Bose particles as a closed thermodynamic system - which is thermally isolated. Then only adiabatic changes in its state are possible in it, i.e. adiabatic expansion (including into the void).

According to the results of the previous sections, in this case, the relic radiation is transferred by a photon gas flow. As we have shown above, the flow rate of photon gas particles varies in the range $0 \leq \vartheta \leq \vartheta_{max}$.

Since, according to SRT, all directions of space are equal, we can assume that the adiabatic expansion occurs radially. Due to the fact that the emitted photons and the speed of the photon gas flow are directed in the same direction, a natural red shift of the CMB wavelengths occurs.

In fact, the time determined by the formula

$$t = \frac{r}{\vartheta}$$

means the time interval during which the disturbance (or “signal”) travels distances r from the moment it was sent. In our case$\vartheta_{max} = \vartheta_{ph}$ is exact and does not depend on experimentally determined empirical parameters.

Moreover, if the speed of "escape" of galaxies increases by only 71 $\frac{km}{sec}$ for colossal distances of one Mega p.s. - then the speed and acceleration of the "expansion" of the Universe is actually equal to zero.

If the characteristic time of establishing the maximum flow velocity is the time of "formation" of the cosmic microwave background radiation with an average energy $\bar{\varepsilon} = 2{,}7\ kT$, then according to the Heisenberg uncertainty relation, we obtain

$$t \approx \frac{h}{\bar{\varepsilon}} = 10^{-12} s.$$

Then the characteristic acceleration of the flow of photon gas particles will be equal to

$$a = \frac{\vartheta_{max}}{t} \approx 1{,}3 \cdot 10^{20} \frac{\text{м}}{\text{c}^2}.$$

Note that approximately the same will be the acceleration of ions in a relativistic plasma - formed in an inertial laser-controlled thermonuclear fusion.

According to the results of [41], the time scale factor is determined by the following formula

$$t = \frac{1}{H} \cdot \left(1 - \frac{2}{3\gamma}\right),$$

here H – is the above given value of the “Hubble constant”, γ – is the adiabatic index of the QRIG inhabiting the Universe. If we assume that the Universe was formed near the temperature of absolute zero - as shown above, then $\gamma \to \infty$ - since, at $T \to 0$, $c_V \to 0$. However, in this case the flow rate of the QRIG is zero (see Figure 5 and formula 86-B), which means $H = 0$. In other words, the age of the universe is infinite. Moreover, any cosmological theory will eventually have to use the equation of state of the gaseous universe. In particular, the values $\gamma: 0, 1, \frac{4}{3}$ used in the cited work are not related to reality, since, as it was shown above, for all really existing QRIG $\gamma \geq \gamma_{ph} > \frac{4}{3}$.

Conclusions and discussions

3.1 Some consequences of the relativistic invariant theory of classical and quantum ideal gases

As emphasized at the beginning of the article: the history of the RQIG theory spans more than a century of time (see, for example, [10, 11, 12, 19, 31, 39]). But there are still a number of open problems awaiting their solutions, some of which are considered in this paper. Let's list them in order:

1. Is the temperature invariant or does it change when moving from one inertial frame of reference to another?

We have proved that, due to the relativistic invariance of the particle momentum distribution function, the temperature is also necessarily invariant. Otherwise, under Lorentz transformations, the DF also changes, i.e. the statistical nature of the system, which leads to the dissimilarity of the equations of state (connecting the macroscopic characteristics of the gas and the temperature) of the RQIG, which contradicts the invariance of the laws of nature in all IFRs.

2. For the first time, analytical expressions for macroscopic characteristics and equations of state for a set of RQIG of particles, antiparticles, and a photon, obeying the canonical distributions of BE and FD, are obtained for arbitrary final values of the ratios of the chemical potential and rest energy to thermal energy based on quantum averaging, without using the thermodynamic potential method.

3. It is shown that the condition for the condensation of Bose particles is $\mu(T_c) = mc^2$. The found difference between the average density of the numbers of the part and antiparticles $n - \bar{n}$ (which some authors call the "number of particles", but we do not adhere to this point of view), shows that if $\mu > 0$, then particles dominate, and for $\mu < 0$, antiparticles prevail (see expression (70)).

Further, taking into account the requirements of statistical and thermal equilibrium, which is possible only if the temperature and chemical potential are the same for the entire system, the chemical potential is equal to zero, since the photon has $\mu_\gamma = 0$, then for all RQIG of particles and antiparticles $\mu = 0$.

For comparison, the thermodynamic equilibrium condition is given based on the thermodynamic potential method based on the canonical Gibbs distribution (which we do not use anywhere), which requires a minimum free energy for each kind of particles. This requirement also necessarily leads to the condition $\mu = 0$.

4. On the basis of the relativistic Bernoulli equation, the first law of relativistic thermodynamics for an adiabatic process, which in the language of entropy means its invariance [31], an expression was obtained that makes it possible to determine the velocity of the flow of RQIG particles.

Based on the analysis of possible values of pressure (p>0) and particle flow velocity (v<c), a fundamental conclusion was obtained that fermions with zero mass cannot exist, since their presence leads to results that are meaningless from a physical point of view.

5. The relativistic quantum gas model of the Universe is considered, in which at absolute zero temperature all particles are in the state of "pure" condensate.

Assuming that at a certain "critical" temperature, due to quantum and thermal fluctuations, there is a continuous production of a particle-antiparticle pair - an electron neutrino and an antineutrino - and the annihilation of this immobile pair into two photons, we conclude that the average number density of produced photons of the relict radiation is exactly equal to the average number density of the annihilated neutrino-antineutrino pair, since $\mathrm{n} + \bar{\mathrm{n}} = 2n_{ph}$. Since for $\mu = 0, n = \bar{\mathrm{n}}$, then $n = n_{ph}$.

6. The conclusion in statistical and thermal equilibrium actually means the invariance of the average macroscopic characteristics over time and their equality to the statistical average, which is possible only at zero chemical potential for all gases at any temperature. The authors of [26, 27, 31] also come to the conclusion that "... both types of bosons (Bose particles and antiparticles) will be more stable at the lowest values of free energy at all temperatures for a fixed "number of particles" (i.e., the difference between the average densities of particles and antiparticles in our interpretation)".

7. Based on numerical simulations, the authors of [27, 31] analyzed the process of the birth of particles and antiparticles that make up an electrically neutral system and came to the following conclusion: "... in the essentially nonrelativistic limit, i.e. at $kT \ll mc^2, \mu$ particles and antiparticles are "suppressed" by the exponential factor $\exp\left(-\frac{mc^2}{kT}\right)$... charge and their annihilation in a vacuum. However, this does not change their total charge, i.e. the condition of electrical neutrality is not violated."

In particular, from expression (70), when $\mu(T_c) = mc^2$, but $kT \ll mc^2$, i.e. at ultralow temperatures, we get [4, 42]

$$n - \bar{n} \approx 4\pi g_1 \sqrt{\frac{\pi}{2}} \left(\frac{\sqrt{mkT}}{h}\right)^3 \left(1 - e^{-2b}\right).$$

It follows from here that the contribution of antiparticles is indeed negligible near the absolute zero temperature, i.e. "suppressed" by the exponential factor $e^{-2b} \ll 1$, in full accordance with the result of numerical simulation in [27, 31].

In addition, using the found expression for the difference between the average densities of the numbers of particles and antiparticles, it is possible to determine the electromagnetic shift of the critical condensation temperature ΔT_c, due to the interaction of charged particles and antiparticles, based on the relation [27]:

$$\Delta T_c = \frac{mc^2}{k} \alpha \frac{\Delta n}{\left(\frac{mc}{h}\right)^3}, \qquad \alpha = \frac{e^2}{2hc}.$$

Here: α – is the so-called fine structure constant, $\Delta n = n - \bar{n}$.

Therefore, as expected, at ultralow temperatures, near which Bose-Einstein condensation takes place, the correction to T_c is very small, since $\alpha \approx \frac{1}{137}$ and $\frac{kT}{mc^2} \ll 1$.

In other words, particles and antiparticles are always born and destroyed only in pairs. In these processes, the law of conservation of electric charge is always fulfilled, i.e. the electrical neutrality of the system remains unchanged, which, as was proved above, is carried out only at zero chemical potential, i.e. in an equilibrium state. This condition leads to the equality $n = \bar{n}$.

Thus, the results obtained by us in a purely analytical way are fully confirmed by the results of numerical simulation in [27, 31], which proves their reliability.

8. It is shown that the particle flow rate of the QRIG cannot exceed

$v_{ph} < v_{s.max}$. Here: v_{ph} and $v_{s.max}$ are, respectively, the speed of the flow of photon gas particles and the maximum speed of sound.

9. It has been established that the adiabatic indices of the actually existing RIGs satisfy the fundamental inequality $\gamma \geq \gamma_{ph} > \frac{4}{3}$, which ensures non-negativity of the pressure and subsonic velocity of their particle flows.

10. The relativistic particle velocity distribution function of the classical RIG MB has found its experimental confirmation [4] (Fig. 1.), which in the energy representation has the form (Fig. 2.)

$$f(\gamma) = \frac{b}{k_2(b)}\gamma\sqrt{\gamma^2 - 1}e^{-b\gamma}, \quad \gamma = \frac{E}{mc^2}, \qquad b = \frac{mc^2}{kT}.$$

11. According to the conclusion of researchers at the CERN Large Hadron Collider, “the process of particle collisions occurs approximately like collisions in an ideal gas of classical relativistic particles” [37, 38], i.e. their spin affiliation can be ignored. This is due to the fact that the accelerated particle beams have a low density, i.e. $n \ll \left(\frac{mc}{h}\right)^3$ and have ultrarelativistic velocities, i.e. $v \approx c$. In this case, the Coulomb interaction between colliding particles can be ignored, since in the situation under consideration $\frac{Ke^2}{dK_BT} \ll 1$. Here: $d = n^{-\frac{1}{3}}$– average characteristic distance between particles; $K_BT \sim TeV$ – thermal energy equivalent to the energy of colliding relativistic particles. For example, for the electron-positron plasma of cosmic rays, the latter ratio is 10^{-23} [36].

Consequently, the representation of an ideal gas under such conditions is quite reasonable and makes it possible to adequately describe the processes under study on the basis of the relativistic distributions of MB, FD, and BE.

We have found an expression for the effective collision cross section σ [4]. It allows one to determine σ for arbitrary, intermediate values of the ratio of the thermal energy and the rest energy of the particle.

$$\frac{\sigma}{\sigma_{max}} = < u >, \ < u > = \frac{<v>}{c}.$$

Here: $\sigma_{max} = \sigma_n \cdot c$, σ_n- is the total geometric cross section of the nucleon, $< v > -$ is the arithmetic mean velocity of the particle colliding with the stationary nucleon[4]:

$$< u > = \frac{2e^{-\frac{1}{x}}}{k_2\left(\frac{1}{x}\right)} \cdot x(1 + x); \qquad x = \frac{kT}{mc^2}$$

It follows from the above graph that even at Planck $T_{Pl} \sim 10^{31}$ K, it is impossible to achieve σ_{max}, because equality $v = c$ is not achieved (fig.3.). Figure 4 shows for comparison a typical dependence of the collision cross section.

12. Based on the above analyses, we will come to the following fundamental conclusions: 1) there can be no infinite compression or “collapse” of the gaseous Universe, which would take place if the inequality $\gamma \leq \frac{4}{3}$ is satisfied (on the other hand, this is a well-known condition instability of the so-called polytropic stars [5]), however, according to the above evidence, all real-life QRIGs have adiabatic exponents $\gamma \geq \gamma_{ph} > \frac{4}{3}$; 2) Analyzing the opposite, we come to the conclusion that there is no irreversible "expansion" of the gaseous Universe, which would take place at a supersonic flow velocity of its particles, i.e. $v > v_{s\,max}$, which is obviously not carried out, since $v \leq v_{ph} < v_{s\,max}$ (Fig. 5.).

Thus, within the framework of the presented relativistic quantum gas model of the Universe, it does not compress and does not scatter, i.e. is in equilibrium and stationary state.

The above proof shows that the gaseous Universe, even in the most critical situation in its evolution, does not shrink to a state with infinite pressure and unlimited average energy density, and its irreversible expansion at supersonic speeds does not occur.

In fact, this is also a proof of its stability, since **the necessary conditions** for thermodynamic equilibrium and the so-called dynamic stability $\gamma > 1$ are simultaneously satisfied [36]. However, we have proved that the **sufficient condition** for its stability is the inequalities $\gamma \geq \gamma_{ph} > \frac{4}{3}$ and $v \leq v_{ph} < v_{s\,max}$.

This means that neither the Chandrasekhar limit nor the Schwarzschild radius are implemented in principle, since they were obtained for $\gamma \leq \frac{4}{3}$--lying stars outside the stability region $\gamma \geq \gamma_{ph} > \frac{4}{3}$ i.e. not a single really existing star turns into a "black" hole and there is no "dark" matter in the Universe.

3.2 On New Approaches in Relativistic Thermodynamics and Relativistic Kinetic Theory

Concluding the paper, let us briefly discuss the recently developed approaches in relativistic thermodynamics and relativistic kinetic theory (in particular, in non-inertial reference frames) and their applications to open problems in this field [45-60].

Naturally, as emphasized above, the most discussed and controversial issue is the law of thermodynamic quantities at transition from one frame of reference to another (in particular, for an accelerated moving observer) [47-54, 59-60]. Equally important, and more important, is the problem of invariance (or noninvariance) of the distribution function [49-50, 56-57, 59-60] (both for single-particle and multi-particle) which has not been finally solved yet. The temperature (of a gas or body) in a moving frame of reference (in particular, with acceleration) is also a subject of incessant debates [48-49, 59-60].

Further, given the limited volume of the article, we will focus our attention mainly on the works of Luca Lusanna and her co-authors [58, 59]. This is due to the fact that, in our opinion, they are closest in content to our studies [4-9, 42-44].

Using the relativistic Hamiltonian equation of motion and Liouville's theorem, it is shown that in the absence of an external force the relativistic Boltzmann equation has an equilibrium solution describing the Boltzmann-Juttner distribution. We obtain the latter as an equilibrium stationary solution of the relativistic Fokker-Planck equation for the momentum distribution function [5], while the initial was the relativistically generalized Langevin equation of motion.

Starting from the nonequilibrium relativistic Gibbs ensemble for the distribution function density and assuming that it as a micro-canonical distribution function tends to equilibrium, the authors managed to define the one-particle distribution function as a statistical mean. Further, by investigating transformational properties of this distribution in inertial relativistic reference frames under Poincaré transformations, it is shown that the one-particle distribution function is Lorentz-scalar, i.e. does not change at transition from one inertial reference frame to another.

Note that, generalizing this result for the multi-particle distribution function, we can also show its relativistic invariance for a system of N non-interacting particles, i.e. an ideal gas, since, according to the energy relation, it is equal to the product of one-particle distribution functions [5].

The authors also managed to prove, using the micro-canonical entropy, the Lorentz-scalar temperature, i.e. independence of the micro-canonical temperature from the speed of the inertial reference system, when there is a thermodynamic limit. And the latter means the relativistic invariance of all macroscopic characteristics of both classical and quantum relativistic ideal gas, since they are determined by a single parameter $b = \frac{kT}{mc^2}$ which is the Lorentz-scalar, according to the above mentioned.

Consequently, the results in [47-48, 59] obtained on the basis of the relativistic micro-canonical ensemble agree with our results obtained in a purely statistical way on the basis of finding averages on the distribution function: for the relativistic classical Maxwell-Boltzmann ideal gas on the relativistic rate distribution function; for the relativistic quantum ideal gas - on the relativistic Fermi-Dirac and Bose-Einstein distribution functions.

We believe that taking into account the interaction of particles at a distance and the noninertiality of the reference frame in the theory of relativistic quantum ideal gases of bose and fermi particles has conceptual importance and will be a subject of future research.

Completing the preliminary stage of the "Relativistic invariant theory of classical and quantum ideal gases", we hope that it will find its application in various areas of relativistic accelerator physics, in the processes of collisions of ultrafast thermal neutrons, as well as in laser-controlled inertial relativistic thermonuclear fusion. However, time will tell the rest.

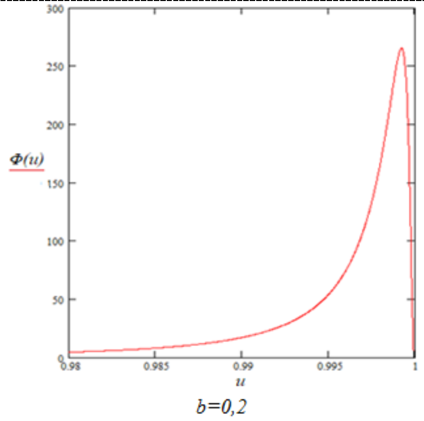

Fig.1. Distribution function of moduli of normalized particle velocities of relativistic ideal gas Maxwell-Boltzmann (b=0.2)

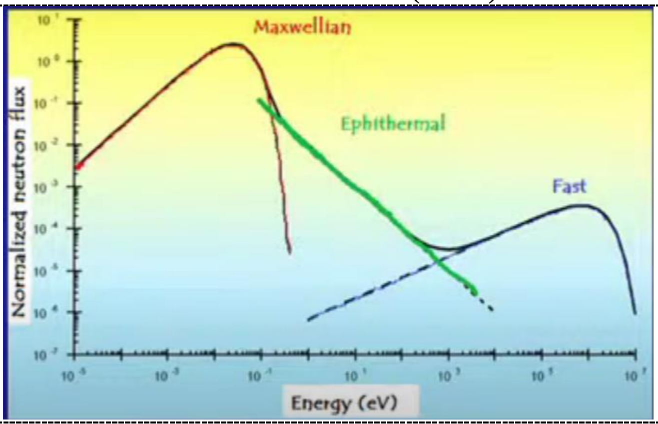

Fig.2. Normalized relativistic energy distribution function (AONSA Neutron school, 21 November, 2022).

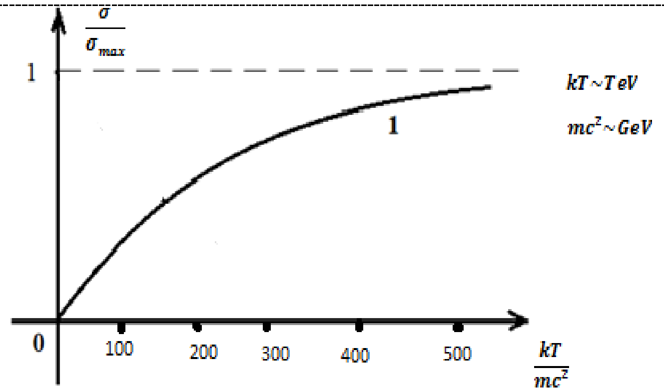

Fig.3. Normalized effective cross section of elastic collision of particles RIG.

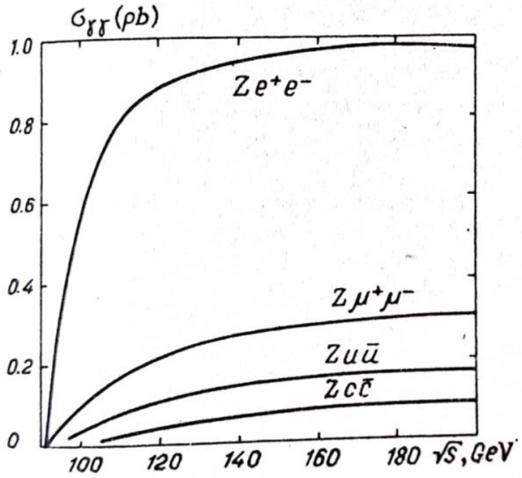

Fig.4. Typical the energy dependence of cross section.

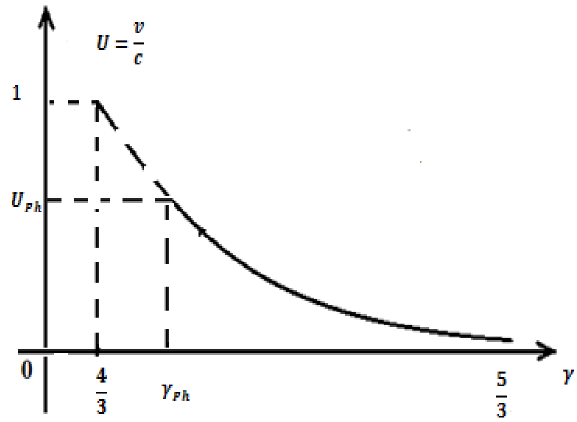

Fig.5. Normalized maximum velocities flow of QRIG.

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