

T-Q Fuzzy subsemiring

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Abstract: In this paper the concept of t- Q Fuzzy subsemiring(normal subsemiring)and t- Q Fuzzy idealsare introduced and some of its properties are discussed. The homomorphic behavior of t- Q Fuzzy subsemiring and t- Q Fuzzy ideals and inverse images are studied.

Keywords: Q Fuzzy set (QFS), Q Fuzzy subsemiring (QFSSR), Q Fuzzy normal subsemiring (QFNSSR), t-Q fuzzy set (t-QFS), t- Q Fuzzy subsemiring (t-QFSSR), t-Q Fuzzy normal subsemiring (t-QFNSSR).

1 Introduction

Using the notation of a fuzzy subset introduced by Zadeh [10], Liu [3] defined fuzzy set and fuzzy ideals of a ring. The notion of fuzzy subgroup was introduced by Rosenfeld [4] in his pioneering paper. Kim [2] studied intuitionistic Q-fuzzy semiprime ideals in semigroups. A Solairaju and R. Nagarajan [9] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. S. Hemalatha, et.al. [1] introduced the concept of Q-fuzzy subring of a ring and established some results. The notion of t-intuitionistic fuzzy subgroups, t-intuitionistic fuzzy quotient group and t-intuitionistic fuzzy subring of a ring have already been introduced by Sharma [5,6,7]. He [8] also introduced the concept of t-fuzzy subring of a ring. In this paper I introduce the concept of t-Q fuzzy subsemiring and t-Q fuzzy normal subsemiring of a ring. Some of its properties are discussed.

2 Preliminaries

Definition 2.1: Let $(R, +, \cdot)$ be a semiring. Let (Q, \cdot) be a non-empty set. A map $A: R \times Q \rightarrow [0, 1]$ is said to be a Q-fuzzy subset of R.

Definition 2.2. Let $(R, +, \cdot)$ be a ring and Q be a non-empty set. A Q-fuzzy subset A of R is said to be a Q fuzzy subsemiring (QFSSR) of R if it satisfies the following conditions:

- (i) $A(x + y, q) \geq \min\{A(x, q), A(y, q)\}$
- (ii) $A(xy, q) \geq \min\{A(x, q), A(y, q)\}$

Definition 2.3. Let $(R, +, \cdot)$ be a ring and Q be a non-empty set. A Q-fuzzy subsemiring A of R is said to be a Q fuzzy normal subsemiring (QFNSSR) of R if $A(xy, q) = A(yx, q)$

Definition 2.4. Let A be a fuzzy set of a ring R. Let $t \in [0, 1]$. Then the fuzzy set A^t of R is called the t-fuzzy subset (t-FS) of R w.r.t. fuzzy set A and is defined as $A^t(x) = \min\{A(x), t\}$, for all $x \in R$.

Definition 2.5. Let A be a fuzzy set of a ring R and $t \in [0,1]$. Then A is called t -fuzzy subring (in short t -FSR) of R if A^t is fuzzy subring of R i.e. if the following conditions hold.

1. $A^t(x-y) \geq \min\{A^t(x), A^t(y)\}$
2. $A^t(xy) \geq \min\{A^t(x), A^t(y)\}$, for all $x, y \in R$.

Definition 2.6. Let A^t be a fuzzy subring of a ring R and $t \in [0,1]$. Then A is called t -fuzzy normal subring (in short t -FNSR) of R if A^t is fuzzy normal subring of R i.e. if $A^t(xy) = A^t(yx)$.

Definition 2.7:

Let A be a QFS of a ring R . Let $t \in [0,1]$. Then A is called a

(a) Q Fuzzy Left Ideal (QFLI) of R if

- i) $A(x+y, q) \geq \min\{A(x, q), A(y, q)\}$
- ii) $A(xy, q) \geq A(y, q)$, for all $x, y \in R$ and $q \in Q$.

(b) Q Fuzzy Right Ideal (QFRI) of R if

- i) $A(x+y, q) \geq \min\{A(x, q), A(y, q)\}$
- ii) $A(xy, q) \geq A(x, q)$, for all $x, y \in R$ and $q \in Q$.

(c) Q Fuzzy Ideal (QFI) of R if

- i) $A(x+y, q) \geq \min\{A(x, q), A(y, q)\}$
- ii) $A(xy, q) \geq \max\{A(x, q), A(y, q)\}$ for all $x, y \in R$ and $q \in Q$.

Definition 2.8: Let f be a mapping from Ring X to Ring Y . For any fuzzy set B in Y , we define a new fuzzy set denoted as $f^{-1}(B) = B(f(x), q)$ for all $x \in X, q \in Q$. For any fuzzy set A in X , we define $f(A)$ in Y by

$$f(A)(y, q) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x, q), & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

3. t-Q-Fuzzy Subsemiring

Definition 3.1. Let A be a Q-fuzzy set (QFS) of a ring R . Let $t \in [0,1]$. Then the fuzzy set A^t of R is called the t -Q-Fuzzy subset (t -QFS) of R with respect to QFS A and is defined as $A^t(x, q) = \min\{A(x, q), t\}$ for all $x \in R$ and $q \in Q$.

Result 3.2. Let A^t and B^t be two t -QFS of a ring R .

Then $(A \cap B)^t = A^t \cap B^t$.

Proof. Let $x \in R, q \in Q$ be any element, then

$$(A \cap B)^t(x, q) = \min\{(A \cap B)(x, q), t\} = \min\{\min\{A(x, q), B(x, q)\}, t\}$$

$$= \min \{ \min \{ A(x, q), t \}, \min \{ B(x, q), t \} \} = \min \{ A^t(x, q), B^t(x, q) \}$$

$$= (A^t \cap B^t)(x, q)$$

Hence $(A \cap B)^t = A^t \cap B^t$

Result 3.3. Let $f : X \rightarrow Y$ be a mapping. Let A and B are two QFS of X and Y respectively, then

- i. $f^{-1}(B^t) = (f^{-1}(B))^t$
- ii. $f(A^t) = (f(A))^t$, for all $t \in [0, 1]$

Proof.

$$f^{-1}(B^t)(x, q) = B^t(f(x, q)) = \min \{ B(f(x, q), t) \} = \min \{ f^{-1}(B)(x, q), t \}$$

$$= (f^{-1}(B))^t(x, q)$$

Hence $f^{-1}(B^t) = (f^{-1}(B))^t$

$$f(A^t)(y, q) = \sup \{ A^t(x, q) : f(x, q) = (y, q) \} = \sup \{ \min \{ A(x, q), t \} : f(x, q) = (y, q) \}$$

$$= \min \{ \sup \{ A(x, q) : f(x, q) = (y, q) \}, t \} = \min \{ f(A)(y, q), t \} = (f(A))^t(y, q)$$

Hence $f(A^t) = (f(A))^t$, for all $t \in [0, 1]$.

Definition 3.4. Let $(R, +, \cdot)$ be a ring and Q be a non-empty set. A Q -fuzzy subset A of R is said to be a t - Q fuzzy subsemiring (t -QFSSR) of R if it satisfies the following conditions:

- (i) $A^t(x + y, q) \geq \min \{ A^t(x, q), A^t(y, q) \}$
- (ii) $A^t(xy, q) \geq \min \{ A^t(x, q), A^t(y, q) \}$, for all $x, y \in R$ and $q \in Q$.

Proposition 3.5: If A is a QFSSR of a ring R , then A is called t -QFSSR of R .

Proof. Let $x, y \in R$ and $q \in Q$.

$$\text{Now } A^t(x + y, q) = \min \{ A(x + y, q), t \} \geq \min \{ \min \{ A(x, q), A(y, q) \}, t \} \geq \min \{ \min \{ A(x, q), t \}, \min \{ A(y, q), t \} \} \geq$$

$$\min \{ A^t(x, q), A^t(y, q) \}$$

Hence $A^t(x + y, q) \geq \min \{ A^t(x, q), A^t(y, q) \}$

$$\text{Also } A^t(xy, q) = \min \{ A(xy, q), t \} \geq \min \{ \min \{ A(x, q), A(y, q) \}, t \} \geq \min \{ \min \{ A(x, q), t \}, \min \{ A(y, q), t \} \} \geq \min \{ A^t(x, q),$$

$$A^t(y, q) \}$$

Hence $A^t(xy, q) \geq \min \{ A^t(x, q), A^t(y, q) \}$

Thus A is t -QFSSR of R .

Remark 3.6: The converse of the above proposition 3.5 need not be true as the following example shows.

Example 3.7: Consider the Ring $(Z_4, +_4, \times_4)$, where $Z_4 = \{0, 1, 2, 3\}$ and $Q = \{q\}$. Define the fuzzy set A of Z_4 by

$$A(x,q) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1,3 \\ 0.3 & \text{if } x = 2 \end{cases}$$

Clearly A is not QFSSR of Z_4 . However, if we take $t=0.2$, then $A^t(x,q) = 0.3$, for all $x \in Z_4$ and $q \in Q$. Now A is a 0.3-QFSSR of Z_4 .

Proposition 3.8: Let A be a QFSSR of the ring R. Let $t \leq p$ where $p = \min\{A(x,q) : \text{for all } x \in R, q \in Q\}$. Then A is t-QFSSR of R.

Proof. Since $t \leq p \Rightarrow p \geq t \Rightarrow \min\{A(x,q) : \text{for all } x \in R, q \in Q\} \geq t \Rightarrow A(x,q) \geq t$ for all $x \in R$.

Therefore $A^t(x+y, q) \geq \min\{A^t(x, q), A^t(y,q)\}$ and $A^t(xy, q) \geq \min\{A^t(x, q), A^t(y,q)\}$ hold. Hence A is t-QFSSR of R.

Proposition 3.9: The intersection of two t-QFSSR's of R is also t-QFSSR of R.

Proof. Let $x,y \in R$ and $q \in Q$.

Now $(A \cap B)^t(x+y, q) = \min\{(A \cap B)(x+y, q), t\} \geq \min\{\min\{A(x+y, q), B(x+y, q)\}, t\} \geq \min\{\min\{A(x+y, q), t\}, \min\{B(x+y, q), t\}\} = \min\{A^t(x+y, q), B^t(x+y, q)\} \geq \min\{\min\{A^t(x, q), A^t(y, q)\}, \min\{B^t(x, q), B^t(y, q)\}\} = \min\{\min\{A^t(x, q), B^t(x, q)\}, \min\{A^t(y, q), B^t(y, q)\}\} = \min\{(A^t \cap B^t)(x, q), (A^t \cap B^t)(y, q)\} = \min\{(A \cap B)^t(x, q), (A \cap B)^t(y, q)\}$

Thus $(A \cap B)^t(x+y, q) = \min\{(A \cap B)^t(x, q), (A \cap B)^t(y, q)\}$

Also $(A \cap B)^t(xy, q) = \min\{(A \cap B)(xy, q), t\} \geq \min\{\min\{A(xy, q), B(xy, q)\}, t\} \geq \min\{\min\{A(xy, q), t\}, \min\{B(xy, q), t\}\} = \min\{A^t(xy, q), B^t(xy, q)\} \geq \min\{\min\{A^t(x, q), A^t(y, q)\}, \min\{B^t(x, q), B^t(y, q)\}\} = \min\{\min\{A^t(x, q), B^t(x, q)\}, \min\{A^t(y, q), B^t(y, q)\}\} = \min\{(A^t \cap B^t)(x, q), (A^t \cap B^t)(y, q)\} = \min\{(A \cap B)^t(x, q), (A \cap B)^t(y, q)\}$

Thus $(A \cap B)^t(xy, q) = \min\{(A \cap B)^t(x, q), (A \cap B)^t(y, q)\}$

Hence $(A \cap B)$ is t-QFSSR of R.

Proposition 3.10: Let A be QFNSSR of a ring R, then A is also t-QFNSSR of R.

Proof. Let $x,y \in R$ and $q \in Q$. Then by above proposition A is t-QFSSR of R. Now $A^t(xy, q) = \min\{A(xy, q), t\} = \min\{A(yx, q), t\} = A^t(yx, q)$. Hence A is t-QFNSSR of R.

Definition 3.11:

Let A be a QFS of a ring R. Let $t \in [0, 1]$. Then A is called a

- a) t-Q Fuzzy Left Ideal (t-QFLI) of R if
 - i) $A^t(x+y, q) \geq \min\{A^t(x, q), A^t(y, q)\}$
 - ii) $A^t(xy, q) \geq A^t(y, q)$, for all $x, y \in R$ and $q \in Q$.
- b) t-Q Fuzzy Right Ideal (t-QFRI) of R if
 - i) $A^t(x+y, q) \geq \min\{A^t(x, q), A^t(y, q)\}$
 - ii) $A^t(xy, q) \geq A^t(x, q)$, for all $x, y \in R$ and $q \in Q$.

c) t -Q Fuzzy Ideal (t -QFI) of R if

$$A^t(x+y,q) \geq \min\{A^t(x,q), A^t(y,q)\}$$

i) $A^t(xy,q) \geq \max\{A^t(x,q), A^t(y,q)\}$ for all $x, y \in R$ and $q \in Q$.

Proposition 3.12: If A is QFLI of a ring R , then A is also t -QFLI of R .

Proof. Clearly $A^t(x+y,q) \geq \min\{A^t(x,q), A^t(y,q)\}$ (From Proposition 3.5)

$$\text{Now } A^t(xy,q) = \min\{A(xy,q), t\} \geq \min\{A(y,q), t\} = A^t(y,q).$$

Hence A is t -QFLI of R .

Proposition 3.13: If A is QFRI of a ring R , then A is also t -QFRI of R .

Proof. Clearly $A^t(x+y,q) \geq \min\{A^t(x,q), A^t(y,q)\}$ (From Proposition 3.5)

$$\text{Now } A^t(xy,q) = \min\{A(xy,q), t\} \geq \min\{A(x,q), t\} = A^t(x,q).$$

Hence A is t -QFRI of R .

Proposition 3.14: If A is QFI of a ring R , then A is also t -QFI of R .

Proof. Follows from Proposition 3.12 and Proposition 3.13.

4. Homomorphism of t - Q Fuzzy Subsemiring:

Theorem 4.1: Let $f: R_1 \rightarrow R_2$ be a ring homomorphism from the Ring R_1 into a Ring R_2 . Let B be t -QFSSR of R_2 .

Then $f^{-1}(B)$ is t -QFSSR of R_1 .

Proof. Let B be a t -QFSSR of R_2 . Let $x_1, x_2 \in R_1$ and $q \in Q$. Then

$$\begin{aligned} f^{-1}(B^t)(x_1+x_2,q) &= B^t(f(x_1+x_2),q) = B^t(f(x_1)+f(x_2),q) \geq \min\{B^t(f(x_1),q), B^t(f(x_2),q)\} \\ &\geq \min\{f^{-1}(B^t)(x_1,q), f^{-1}(B^t)(x_2,q)\} \end{aligned}$$

$$\begin{aligned} \text{Now } f^{-1}(B^t)(x_1x_2,q) &= B^t(f(x_1x_2),q) = B^t(f(x_1)f(x_2),q) \geq \min\{B^t(f(x_1),q), B^t(f(x_2),q)\} \\ &\geq \min\{f^{-1}(B^t)(x_1,q), f^{-1}(B^t)(x_2,q)\} \end{aligned}$$

Hence $f^{-1}(B)$ is t -QFSSR of R_1 .

Theorem 4.2: Let $f: R_1 \rightarrow R_2$ be a ring homomorphism from the Ring R_1 into a Ring R_2 . Let B be t -QFNSSR of R_2 .

Then $f^{-1}(B)$ is t -QFNSSR of R_1 .

Proof. Let B be t -QFNSSR of R_2 . Let $x_1, x_2 \in R_1$ and $q \in Q$.

$$\begin{aligned} f^{-1}(B^t)(x_1x_2,q) &= B^t(f(x_1x_2),q) = B^t(f(x_1)f(x_2),q) = B^t(f(x_2)f(x_1),q) = B^t(f(x_2x_1),q) \\ &= f^{-1}(B^t)(x_2x_1,q) \end{aligned}$$

Hence $f^{-1}(B)$ is t -QFNSSR of R_1 .

Theorem 4.3: Let $f: R_1 \rightarrow R_2$ be a ring homomorphism from the Ring R_1 into a Ring R_2 . Let B be t -QFLI of R_2 . Then

$f^{-1}(B)$ is t -QFLI of R_1 .

Proof. Let B be t -QFLI of R_2 . Let $x_1, x_2 \in R_1$ and $q \in Q$.

Clearly $f^{-1}(B^t)(x_1+x_2,q) \geq \min\{f^{-1}(B^t)(x_1,q), f^{-1}(B^t)(x_2,q)\}$ (from theorem 4.1)

Now $f^{-1}(B^t)(x_1x_2, q) = B^t(f(x_1x_2), q) = B^t(f(x_1)f(x_2), q) \geq B^t(f(x_2), q) \geq f^{-1}(B^t)(x_2, q)$

Thus $f^{-1}(B)$ is t-QFLI of R_1 .

Theorem 4.4: Let $f: R_1 \rightarrow R_2$ be a ring homomorphism from the Ring R_1 into a Ring R_2 . Let B be t-QFRI of R_2 . Then $f^{-1}(B)$ is t-QFRI of R_1 .

Proof. It is similar to Theorem 4.3.

Theorem 4.5: Let $f: R_1 \rightarrow R_2$ be a ring homomorphism from the Ring R_1 into a Ring R_2 . Let B be t-QFI of R_2 . Then $f^{-1}(B)$ is t-QFI of R_1 .

Proof. It can be obtained from Theorem 4.3 and Theorem 4.4.

Theorem 4.6: Let $f: R_1 \rightarrow R_2$ be surjective ring homomorphism and A be t-QFSSR of R_1 . Then $f(A)$ is t-QFSSR of R_2 .

Proof. Since A is t-QFSSR of R_1 . Let $y_1, y_2 \in R_2$ and $q \in Q$. Then there exist some $x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

$$\begin{aligned} f(A^t)(y_1+y_2, q) &= (f(A))^t(y_1+y_2, q) = \min\{f(A)((f(x_1)+f(x_2)), q), t\} \\ &= \min\{f(A)(f(x_1+x_2), q), t\} \geq \min\{A(x_1+x_2, q), t\} = A^t(x_1+x_2, q) \\ &\geq \min\{A^t(x_1, q), A^t(x_2, q)\}, \text{ for all } x_1, x_2 \in R_1 \text{ such that } f(x_1) = y_1 \text{ and } f(x_2) = y_2 \\ &\min\{\bigvee\{A^t(x_1, q): f(x_1) = y_1\}, \bigvee\{A^t(x_2, q): f(x_2) = y_2\}\} = \min\{f(A^t)(y_1, q), f(A^t)(y_2, q)\} \end{aligned}$$

Thus $f(A^t)(y_1+y_2, q) \geq \min\{f(A^t)(y_1, q), f(A^t)(y_2, q)\}$

$$\begin{aligned} \text{Now } f(A^t)(y_1y_2, q) &= (f(A))^t(y_1y_2, q) = \min\{f(A)(f(x_1)f(x_2)), q), t\} \\ &= \min\{f(A)(f(x_1x_2), q), t\} \geq \min\{A(x_1x_2, q), t\} = A^t(x_1x_2, q) \\ &\geq \min\{A^t(x_1, q), A^t(x_2, q)\}, \text{ for all } x_1, x_2 \in R_1 \text{ such that } f(x_1) = y_1 \text{ and } f(x_2) = y_2 \\ &\min\{\bigvee\{A^t(x_1, q): f(x_1) = y_1\}, \bigvee\{A^t(x_2, q): f(x_2) = y_2\}\} = \min\{f(A^t)(y_1, q), f(A^t)(y_2, q)\} \end{aligned}$$

Thus $f(A^t)(y_1y_2, q) \geq \min\{f(A^t)(y_1, q), f(A^t)(y_2, q)\}$

Hence $f(A)$ is t-QFSSR of R_2 .

Theorem 4.7: Let $f: R_1 \rightarrow R_2$ be surjective ring homomorphism and A be t-QFNSSR of R_1 . Then $f(A)$ is t-QFNSSR of R_2 .

Proof. Since A is t-QFNSSR of R_1 . Let $y_1, y_2 \in R_2$ and $q \in Q$. Then there exist some $x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Clearly $f(A)$ is t-QFSSR of R_2 (from Theorem 4.6)

$$\begin{aligned} \text{Now } (f(A))^t(y_1y_2, q) &= f(A^t)(f(x_1)f(x_2), q) = f(A^t)(f(x_1x_2), q) = \bigvee\{A^t(x_1x_2, q): f(x_1x_2) = y_1y_2\} \\ &= \bigvee\{A^t(x_2x_1, q): f(x_1x_2) = y_1y_2\} = f(A^t)(f(x_2x_1), q) = f(A^t)(f(x_2)f(x_1), q) = (f(A))^t(y_2y_1, q) \end{aligned}$$

Hence $f(A)$ is t-QFNSSR of R_2 .

Theorem 4.8: Let $f: R_1 \rightarrow R_2$ be surjective ring homomorphism and A be t-QFLI of R_1 . Then $f(A)$ is t-QFLI of R_2 .

Proof. Since A is t-QFNSSR of R_1 . Let $y_1, y_2 \in R_2$ and $q \in Q$. Then there exist some $x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Clearly $f(A^t)(y_1+y_2, q) \geq \min\{f(A^t)(y_1, q), f(A^t)(y_2, q)\}$ (from Theorem 4.6)

Now $(f(A))^t(y_1y_2, q) = \min\{f(A)(f(x_1)f(x_2), q), t\} = \min\{f(A)(f(x_1x_2), q), t\}$
 $= \min\{A(x_1x_2, q), t\} = A^t(x_1x_2, q) \geq A^t(x_2, q) = \min\{A(x_2, q), t\} = \min\{f(A)((f(x_2), q), t\} = \min\{f(A)((y_2, q), t\} = (f(A))^t(y_2, q).$
 Thus $(f(A))^t(y_1y_2, q) \geq (f(A))^t(y_2, q)$

Hence $f(A)$ is t -QFLI of R_2 .

Theorem 4.9: Let $f: R_1 \rightarrow R_2$ be surjective ring homomorphism and A be t -QFRI of R_1 . Then $f(A)$ is t -QFRI of R_2 .

Proof. It is similar to Theorem 4.8

Theorem 4.10: Let $f: R_1 \rightarrow R_2$ be surjective ring homomorphism and A be t -QFI of R_1 . Then $f(A)$ is t -QFI of R_2 .

Proof. It can be obtained from Theorem 4.8 and Theorem 4.9.

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