Applications of Laplace Transformation

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Abstract
Laplace transform is the most effective tool that can be applied in various areas of engineering and science. It helps in solving complex problems with a simple approach just like the applications of transfer functions to solve ordinary differential equations. In physics, its application to electric circuit analysis relation with probability. Difficulter application on Load frequency control in the area of power systems engineering is also talk through.

INTRODUCTION

Laplace transform is an integral transform method which is particularly functional in solving linear ordinary differential equations. It finds immense applications in different areas of physics, electrical engineering and mathematics and so on. The Laplace transform can be transcribed as a transformation from the time province where inputs and outputs are functions of time to the frequency domain where inputs and outputs are functions of complex angular frequency.

Definitions

Unit step function: The unit step function is denoted by $u_a(t)$ and defined by $u_a(t)$ = {0, $t < a$ 1, $t \geq a$

Moment generating function: Suppose X is a continuous random variable (R.V.) with probability density function (pdf) f(x). The moment generating function (mgf) of X is defined as $M_x(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$; t is a parameter.

Laplace Transform: Areal valued function f(t) defined for all $t \geq 0$ then Laplace Transform of f(t), represented by □(f(t)) is defined by F(s) = □$(f(t)) = \int_0^{\infty} e^{-st} f(t)dt$; t is real, s may be either real or complex provided integral $\int e^{-st} f(t)dt$ converges i.e. $\int |e^{-st} f(t)|dt < \infty$

Inverse Laplace Transform: If □$(f(t))$ = F(s) then □$^{-1}$(F(s)) = $f(t)$, $t \geq 0$

Characteristic function: If f(t) be a function defined on real line
then characteristic function of f (C.F.) = $\int_{-\infty}^{\infty} e^{iyt} f(t)dt$ provided $\int_{-\infty}^{\infty} |f(t)|dt < \infty$

Some Important Properties of LaplaceTransforms

(a) If □$(f(t))$= F(s) and □$(g(t))$= G(s) then □(αf(t)+ βg(t)) = αF(s) + βG(s)
(b) Let □ (f(t)) = F(s) then □ (e^{at}f(t)) = F(s-a)
(c) If □$(f(t))$= F(s) then □(f(t-λ)) = $e^{-s\lambda}$F(s)
(d) If □$(f(t))$= F(s) then □(f(at)) = $\frac{1}{a}F(\frac{s}{a})$
(d) Let f(t) be piecewise continuous and of exponential order over each finite interval and □(f(t))=F(s); F(s) is differentiable and F'(s) = □{-tf(t)} moreover □{t^n f(t)}= $(-1)^n \frac{d^n}{ds^n}$F(s)
(e) Let f,f', -- - - $f^{(n-1)}$ be continuous on [o,□) and of exponential order and □(f(t)) = F(s). if $f^{(n)}(t)$ be piecewise continuous on [o,□), then □{$f^{(n-1)}(t)$} = s^n F(s) – s^{n-1}f(0) -s^{n-2}f'(o) - … -$f^{(n-1)}(0)$

Probability Theory

Laplace transform and Expected Value

The Laplace transform is defined as an expected value. If X is a continuous random variable with probability density function f, then the Laplace transform of fis given by $\square(f(t)) = \int_0^\infty \; e^{-st}f(t)dt$ = E(e^{-sT})

Laplace Transform and Characteristic Function

Let f(t) be defined on [0, ∞) and $\int_0^\infty \; e^{-xt}f(t)dt$<∞ for some x ≥ 0 then characteristic function of e^{-xt}f(t)

i.e. C.F. = $\int_0^\infty \; e^{-iyt}e^{-xt}f(t)dt$ = $\int_0^\infty \; e^{-(x+iy)t}f(t)dt$

= $\int_0^\infty \; e^{-st}f(t)dt$ where s = x+iy and Re(s) ≥ 0

C.F. of e^{-xt}f(t) = $\square(f(t))$

Laplace Transform and Moment Generating Function

If all the moments exist then $\square(f(t)) = \int_0^\infty \; e^{-st}f(t)dt$

$$= \int_0^\infty \; \{\sum_{n=0}^\infty \; (-1)^n \frac{s^n t^n}{n!}\} f(t)dt$$

$$= \sum_{n=0}^\infty \; (-1)^n \frac{s^n}{n!} (\int_0^\infty \; t^n f(t)dt)$$

$$= \sum_{n=0}^\infty \; (-1)^n \frac{s^n}{n!} m_n \; ; \; m_n = E(T^n)$$

Ordinary Differential Equation

Initial Value Problem

Let the initial value problem $y'' + y = 1$, y(0) = y'(0) = 1

Taking Laplace transform on both sides, we get $\square(y'') + \square(y) = \square$ (1)

i.e. s$^2\square(y)$ – s y(0) - y'(0) + $\square(y)$ = $\frac{1}{s}$

i.e. $(s^2 + 1)\,\square(y) = \frac{1}{s}$

i.e. $\square(y) = \frac{1}{s(s^2+1)}$

i.e. Taking Inverse Laplace transform on both sides, we get y = $\square^{-1}(\frac{1}{s(s^2+1)})$

i.e. y = $\square^{-1}(\frac{1}{s} - \frac{s}{s^2+1})$ = 1 – cost

Boundary Value Problem

Consider the boundary value problem $y'' + \lambda^2 y =$ cosλt, y(0) = 1,y($\frac{\pi}{2\lambda}$) = 1

Taking Laplace transform on both sides, we get $\square(y'') + \lambda^2\square(y) = \square$ (cosλt)

i.e. s$^2\square(y)$ – s y(0) - y'(0) + $\lambda^2\square(y)$ = $\frac{s}{s^2+\lambda^2}$

i.e. $(s^2 + \lambda^2)\,\square(y) = \frac{s}{s^2+\lambda^2}$ + s y(0) +y'(0)

i.e. $\square(y) = \frac{s}{(s^2+\lambda^2)^2} + \frac{sy(0)}{s^2+\lambda^2} + \frac{y'(0)}{s^2+\lambda^2}$

i.e. Taking Inverse Laplace transform on both sides, we get y = $\square^{-1}(\frac{s}{(s^2+\lambda^2)^2} + \frac{sy(0)}{s^2+\lambda^2} + \frac{y'(0)}{s^2+\lambda^2})$

i.e. y = $\frac{1}{2\lambda}$t sinλt + cosλt + $\frac{y'(0)}{\lambda}$ sinλt

also 1 = y($\frac{\pi}{2\lambda}$) = $\frac{\pi}{4\lambda^2} + \frac{y'(0)}{\lambda}$ i.e. $\frac{y'(0)}{\lambda}$ = 1 - $\frac{\pi}{4\lambda^2}$

i.e. y = $\frac{1}{2\lambda}$t sinλt + cosλt + (1 - $\frac{\pi}{4\lambda^2}$)sinλt

Physics

In nuclear physics, the following fundamental relationship governs radioactive decay: the number of radioactive atoms 'n' in a sample of a radioactive isotope decays at a rate proportional to N. This leads to the first order linear differential equation $\frac{dn}{dt} = -\lambda n$ where λ is the decay constant.

Solution: Given differential equation is $\frac{dn}{dt} = -\lambda n$

i.e. $\frac{dn}{dt} + \lambda n = 0$

Taking Laplace transform on both sides, we get (s N(s) – n(0)) + λN(s) = 0; N(s) = □$(n(t))$

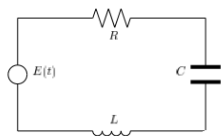
i.e. N(s) = $\frac{n(0)}{s+\lambda}$

Now take Inverse Laplace transform on both sides, we get n(t) = □$^{-1}(n(t))$ = □$^{-1}\left(\frac{n(0)}{s+\lambda}\right)$ = n(0)$e^{-\lambda t}$

Which is right form for radioactive decay.

Electrical Circuits

In the (RCL) circuit in Figure, the voltage drops across the inductor, resistor, and capacitor are given by L(dI/dt), RI, and(1/C), $\int_0^t I(\tau)d\tau$, respectively, where Kirchoff's voltage law states that the sum of the voltage drops across the individual components equals the impressed voltage, E(t), that is, L$\frac{dI}{dt}$ + RI + $\frac{1}{C}\int_0^t I(\tau)d\tau$ = E(t)



L: inductance (constant)
R: resistance (constant)
C: capacitance (constant)
I: current

Now suppose that the current I in an electrical circuit satisfies

L$\frac{dI}{dt}$ +RI = E_0 sinωt; L, R, E_0, and ω are constants. Find I(t) fort > 0if I(0) = 0

Solution: Given that L$\frac{dI}{dt}$ +RI = E_0sinωt

Taking Laplace transform on both sides, we get L(s $L(I) -$ I(0)) + R $L(I)$ = $\frac{E_0 s}{s^2+\omega^2}$

i.e. (Ls+R) $L(I)$ = $\frac{E_0 s}{(s^2+\omega^2)}$ i.e. □(I) = $\frac{E_0 s}{(Ls+R)(s^2+\omega^2)} = \frac{\frac{E_0 L\omega}{R^2+L^2\omega^2}}{s+\frac{R}{L}} + \frac{\left(\frac{-E_0 L\omega}{R^2+L^2\omega^2}\right)s+\frac{E_0 R\omega}{R^2+L^2\omega^2}}{(s^2+\omega^2)}$

i.e. I =□$^{-1}\left(\frac{\frac{E_0 L\omega}{R^2+L^2\omega^2}}{s+\frac{R}{L}} + \frac{\left(\frac{-E_0 L\omega}{R^2+L^2\omega^2}\right)s+\frac{E_0 R\omega}{R^2+L^2\omega^2}}{(s^2+\omega^2)}\right) = \frac{E_0 L\omega}{R^2+L^2\omega^2}e^{-\frac{R}{L}t} + \frac{E_0 R\omega}{R^2+L^2\omega^2}\sin\omega t - \frac{-E_0 L\omega}{R^2+L^2\omega^2}\cos\omega t$

Conclusion

The paper presented the application of Laplace transforms in various areas of physics and electrical power engineering. Apart from, Laplace transform is the most effective mathematical tool to simplify complex problems in the area of stability and control. Laplace Transforms have become an essential part of modern science, being used in a large number of different disciplines. Whether they are being used in electrical circuit analysis, signal processing, or even in modeling radioactive decay in nuclear physics, they have quickly gained popularity among the intellectual community that deals with these subjects on a day to day basis. With the ease of application of Laplace transforms in myriad of scientific applications, many research software's have made it.

References

- D. Poularikas, The Transforms and Applications Handbook (McGraw Hill, 2000), 2nd ed.
- Joel L.Schiff , " The Laplace Transform Theory and Applications " Springer, New York,1991.
- H.K.Dass "Advanced Engineering Mathematics" S.Chand &company Limited,New Delhi, 2009.
- K. Riess, American Journal of Physics 15, 45 (1947).
- M.J.Roberts, Fundamentals of Signals and Systems (McGraw Hill, 2006), 2nd ed.
- Sarina Adhikari, Department of Electrical Engineering and Computer Science, University of Tennessee.