First Order Ordinary Differential Equation-Based Models of Bio-Sciences and its Application: Case of Study Adigrat University, Ethiopia

Gebrehiwot Gebreyesus Kahsay (MSc.)

Department of Mathematics, Adigrat University, Adigrat, Ethiopia

Abstract: World of mathematics concept, where a model is established. We then manipulated the model using techniques or computer aided numerical computations. finally, we then investigate the model that will lead us to the solution to our mathematics problems related to the model. Which is translated in to a real life problem. Mathematical modeling has its place in all sciences and hence the paper concerns mathematical models in biosciences (population biology), and hence this paper will have intended as an introduction to the formulation, analysis and application of mathematical models that describe the dynamics of biological populations. Over the last hundred years' technique have been developed for the solution of first order differential equations while quite a major portion of the techniques is only useful for academic purposes, there are some which are important in the solution of real problems arising from science and engineering. The application of first ordinary differential equation in bioscience in particular on population model or size of population problem will study the method of separation of variable. Where we use the methods to find the solution of the population problem. That requires the use of first order differential equation and these solutions are very useful in mathematics, biology physics especially in dealing with the problems involving size of population that requires the use of model population dynamical system.

Key words: Differential equations, population dynamics, method of separable variable.

1. Introduction

In mathematics a first ordinary differential equation is an equation that contain only first derivative and it has many applications in mathematics, biology, physics engineering and other subjects. Because any physical laws and relations appear mathematically in the form of such equations. In some of the application that are in mathematics, a first order differential equation plays a prominent role in biology that including model of population dynamical systems, biochemical reaction and organism growth. According to some historians of mathematics the study of differential equations began in 1675.when Gottfried Wilhelm von Leibniz wrote the equation.

$$\int x dx = \frac{1}{2}x^2 \qquad (1)$$

The research for general methods of integrating differential equation began Isaac newton classified first order differential equation in to three classes.

$$\frac{dy}{dx} = f(x), \quad \frac{dy}{dx} = f(x, y), \quad \text{and } x\frac{dv}{dx} + y\frac{dv}{dy} = v \qquad (2)$$

The first two classes contain only ordinary derivatives to single one or more dependent variables with respect to single independent variable and are known today as ordinary differential equations. Difference equations and iterative maps occur naturally in mathematical biology. An important problem is how the population size of a given species, for example dividing cells or bacteria, varies from one-time point to another time point. Let $N(t)$ be the population of a species at time t and $N(t+1)$ the population at time $t+1$. The change in population size during the interval between these times is given by the following growth equation, also known as the logistic map:

$$N(t+1) = rN(0)(1 - N(t))$$

where $N(0)$ represents the initial population at time 0, r is a positive number corresponding to an overall growth rate, and the last negative term represents increased competition as the population grows (over limited shared resources for example).This paper will have intended as an introduction to the formulation, analysis and application of mathematical models that describe the dynamics of biological populations. If we consider a population to be the collection of individuals of a particular species that lives within a well- defined area, any changes in the number of individuals within this population comes about by reproduction (birth), death or migration of individual organisms. The simplest model was proposed still in1798 by British scientist Thomas Robert Malthus. this model reflects exponential growth of population and can be described by the differential equation $\frac{dN(t)}{dt} = rN(t)$, where $N(t)$ denotes population size at time t and r is the growth rate (Malthusian parameter) the derivative $\frac{dN(t)}{dt}$ the rate of change of $N(t)$ and t itself can be any positive number. Most notably, I have used the book by Horst R., Thieme (2003), Andre M., de Roos and Edelstein-Keshet (1988) as sources for the paper presented. However, all these books have slightly different approaches and put emphasis on slightly different aspects of theoretical biology. And hence the main intention of this paper is to emphasize more the biological and the conceptual aspects of the theory in that to the formulation of models (the model building stage) and the interpretation of the mathematical analysis in terms of biological conclusions. Plato's allegory of the cave suggests that we only see the shadows of reality; or, following Kant, that we only see phenomena rather than the noumena. St. Paul (first letter to the Corinthians), puts it this way we obtain our knowledge in parts, and we prophesy in parts. St Paul declares our knowledge patchwork, and Immanuel Kant distinguishes between the phenomenon we can recognize and underlying noumenon which remains concealed. In more modern terms, we only see reality through models, first through the spontaneous models created by our senses, then through the deliberate models of science and arts, and finally through the meta- models of philosophy and theology In this scenario, mathematics has its place as a form of symbolic modeling, namely, representation and analysis of reality through mathematical symbols and concepts. Biology, the science of life has developed its own (nonmathematical) models. But recently the formulation of the dynamics of biological populations in mathematical equations the analysis of those equations and the reinterpretation of the results in biological terms has become a valuable source of insight, a selected sample of which has been collected in this paper. A treatise in mathematical biology can be organized according to biological topics or mathematical concepts and tools. this paper organizing with unconstrained population growth for single species and classical models of density dependent population growth for single species. A course in elementary differential equation may be helpful, but the methods of integrating factors, variation of parameters, separation of variables, and transformation of variables are developed from scratch and applied to more examples and practiced in more theorems than in typical textbook for elementary differential equations.

If we look at a single species exclusively, the population rates in the fundamental balance equation of population dynamics would only depend on the size of the respective population N, and on the time variable t. Although in most case the dynamics of this population cannot be isolated from the dynamics of others it is still useful to look at a single species dynamic for a number of reasons.

- ✓ Single species dynamics occur in laboratory environment.
- ✓ Sometimes it may be possible to blend other species into the environment.
- ✓ Understanding one species dynamics helps us to understand the more complex multi species dynamics.

There are two separate cases to consider unconstrained population growth. Where the per capita rates depend only on environmental conditions, i.e. They are functions of time only or are just constant; and density- dependent population growth, where the per capita rates are affected by the population size, N. The first case occurs if a species has unlimited resources and is not subject to predators or competitors; the per capita rates are then just functions of time t or may even be

1.1 Objectives of the Study

- The aim and objective of this paper is to use first order differential equation in solving some problems that are in population dynamics.

- The objective of this paper is to obtaining simple approximation formula which have a suggestive biological interpretation in the context of first order differential equation.
- Mode exponential growth with first order differential equation and solving exponential growth problem.

1.2 Significant of the Study

Hopefully the output of the reviews will help:

- To introduce differential equations (first order differential equations) as simulation tools in biological sciences.
- To investigate application of first order ordinary differential equation in real world problem particularly in population dynamic.
- To determine growth rate of open and closed population using first order ordinary differential equations.
- To provide a non-technical review of a well-established modeling platform, namely differential equations, that harnesses the powerful tools of calculus to analyze the time-dependent behavior of dynamical systems.

1.3 Scope of the Study

- The scope of this paper is to give an insight into the application of first order differential equation in model of population dynamics and is limited to the first order differential equation.

2. First Order Differential Equation

A differential equation is an equation that involves one or more derivates of differentials that is any equation containing differential coefficients is called a differential equation. It is also defined as an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.

Ordinary differential equation is an equation which involves ordinary derivatives of one or more dependent variables with respect to single independent variable.

The order of the differential equation is the order of the highest derivative that occurs in the equation. For example, the equation

$$\frac{d^3y}{dx^3} + \cos x \frac{dy}{dx} = xy \text{ and } \frac{dy}{dx} + 3y = 2x \quad \ldots\ldots(3)$$

are differential equation of degree 3 and 1 respectively.

The degree of a differential equation is the degree of the highest order derivative present in the equation after the differential equation has been made free from the radicals and radicals as far as the derivatives are concerned. For example, the degree of the differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 6y + 2 = 0 \quad \ldots\ldots (4)$$

Is 1.

2.1 Linear and Non-linear Differential Equation

A differential equation is said to be linear if and only if the following conditions are satisfied

i. The dependent variable and its derivatives are all first degree that is the power of each term involving dependent variable is one.
ii. There is no term involving product of the dependent variable and all its derivatives.
iii. There is no transcendental functions involving the unknown function.

Differential equation is said to be non-linear if all the above three given properties are not satisfied.

For example, the differential equation

$$\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+2y=\cos x \text{ and } y\frac{d^2y}{dx^2}-2\frac{dy}{dx}=x+\cos y \text{ ....(5)}$$

are linear and nonlinear ordinary differential equation respectively.

2.2 Initial and Boundary Conditions

A differential equation with additional conditions on the solution at exactly one point is called initial value problem (IVP). The conditions are called initial conditions. If the conditions are given at two or more point it is called a boundary value problem (BVP) and the conditions are called boundary conditions. For example:

$$\frac{dy}{dx}+y=3, \; ; y(0)=2 \text{ and } \frac{d^2y}{dx^2}-\frac{dy}{dx}=x^3 \; ; \; y(0)=4 \; \& \; \frac{dy(1)}{dx}=-2 \text{ ....(6)}$$

are a first order initial value problem and second order boundary value problem respectively.

2.3 Solution of Differential Equation

A solution of a differential equations is a relation between the dependent and independent variables involving the derivatives such that thus relations and the derivatives obtained from it satisfies the given differential equation.
There are two types of solution those are general and particular solutions.

i. A solution which contains a number of arbitrary constants equal to the order of the differential equations is called general solution.
ii. A solution free from arbitrary constant which is obtained from a general solution by giving a particular value to the arbitrary constants is called particular solution.

The first order differential equation has a variety of methods in finding the solution of equation which include.

a. Variable Separable
b. Linear first order differential Equations
c. Equation Reducible to variable separable
d. Homogenous and non-homogenous Equations.
e. First Order Exact, non-exact and integral factors of Differentia Equations

a) Variable Separable

The first order ordinary differential equation of the form

$$\frac{dy}{dx}=H(x,y) \text{ ....(7)}$$

is called separable provided that $H(x,y)$ can be written as the product of x and y. That is $\frac{dy}{dx}=g(x)h(y)=\frac{g(x)}{f(y)}$ where $h(y)=\frac{1}{f(y)}$ and this implies that equation (7) becomes

$$f(y)dy=g(x)dx \text{ .... (8)}$$

In principle it is easy to solve this special types of differential equation simply by taking integration both sides, Then the general solution is

$$\int f(y)dy=\int g(x)dx+c \text{ .... (9)}$$

Where c is an arbitrary constant.

b) Linear First Order Differential Equation

a linear first order differential equation is an equation that can be expressed in the form of

$$\frac{dy}{dx}+p(x)y=q(x) \quad \text{.......} (10)$$

Where $p(x)$ and $q(x)$ are constant or any given continuous functions of x. Hence that he general solution of equation (10) Using an integrating factor has the form

$$y=\frac{1}{\mu(x)}\left[\int q(x)\mu(x)dx+c\right] \quad \text{.......} (11)$$

where $\mu(x)=e^{\int p(x)dx}$ is integral factor

3. Applications First Order Differential Equation in Bio-Sciences

Mathematical modeling first concentrates on one population and expresses the change of its size in terms of concepts like birth rates, mortality rates, and emigration and immigration rates which take the form of either population rates or per capita rates.

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable t will represent time. The units of time can be hours, days, weeks, months, or even years. The variable N will represent size of population. Since the population varies over time, it is understood to be a function of time.

If we consider a species of mammal; we may like to have the population size to be given in numbers of individuals. Then the size can change by four processes: births, deaths, emigration, and immigration. If there is no emigration or immigration, we speak of a closed population, otherwise of an open population. The fundamental balance equation of population dynamics takes these four processes in to account by,

$$N'(t)=B(t)-D(t)+I(t)-E(t) \quad \text{.......} (12)$$

Where, $N'(t)$ is the rate of change of population size $N(t)$ at time t, $B(t)$ denotes the population birth rate, $D(t)$ the population death rate, $I(t)$ the immigration rate and $E(t)$ the emigration rate. More precisely, $B(t)$ is the number of births per time unit, at time t, $D(t)$ is the number of deaths per unit of time t.

As for births, deaths, and emigration, it is also makes sense to speak about per capita rates. the following relations hold between population rate and per capita rates:

$$B(t)=\beta(t)N(t), \quad D(t)=\mu(t)N(t), \quad \text{and} \quad E(t)=\eta(t)N(t) \quad \text{.......} (13)$$

Where $\beta(t),\mu(t)$ and $\eta(t)$ are the per capita birth rate, death rate and emigration rate respectively

Note: It makes a little sense in this context to speak immigration rate; for it may well happen that $N(t)=0$ and $I(t)\succ 0$, if a species invades a formerly empty habitat.

- ❖ If there is neither immigration nor emigration, equation (12) takes the form of a linear ordinary differential equation,

$$\left.\begin{aligned} N' &= \rho(t)N, \\ \rho(t) &= \beta(t)-\mu(t) \end{aligned}\right\} \quad \text{.......} (14)$$

which is an equation of a first order linear homogenous differential equation for closed population. The difference ρ is sometimes called the intrinsic (per capita) growth.

Equation (14) can be solved by a method called separation of variables, where we collect all terms containing the dependent variable N on one side of the equation we have:

$$\Rightarrow \frac{N'(t)}{N(t)} = \rho(t).$$

We integrate both sides with respect to s and use the fundamental theorem of calculus,

$$\ln N(s)\Big|_{t_0}^{t} = \int_{t_0}^{t} \rho(s)ds,$$

$$\ln N(t) - \ln N(t_0) = \int_{t_0}^{t} \rho(s)ds$$

$$= \int_{0}^{t} \rho(s)ds - \int_{0}^{t_0} \rho(s)ds$$

Exponentiation both sides provides

$$\left.\begin{aligned} N(t) &= N_0 \frac{Q(t)}{Q(t_0)}, \\ Q(t) &= \exp\left(\int_0^t \rho(s)ds\right) \end{aligned}\right\} \ldots\ldots (*)$$

Hence equation (*) is the general solutions of equation (14) where N_0 is the population size at some time t_0 and Q can be chosen as the exponential of any anti-derivative of ρ.

If the per capita birth and death rates are constant, so is $\rho(t) = \overline{\rho}$, where $\overline{\rho} \succ 0$ or $\overline{\rho} \prec 0$ and we obtain its general solution as,

$$\frac{N'(t)}{N(t)} = \overline{\rho}ds \ldots\ldots (15)$$

Integrating both sides with respect to s on the interval $[t_0, t]$ we get;

$$\ln N(S)\Big|_{t_0}^{t} = \int_{t_0}^{t} \overline{\rho}ds$$

$$\ln N(t) - \ln(t_0) = \overline{\rho}s\Big|_{t_0}^{t} \ldots\ldots (16)$$

Equation (16) is equivalence to,

$$N(t) = N(t_0)e^{\overline{\rho}(t-t_0)} \ldots\ldots (**)$$

This is the notorious exponential growth, in which a population doubles (or halves) its size in a fixed amount of time if $\overline{\rho} \succ 0$ or ($\overline{\rho} \prec 0$ respectively).

Example : A population of small town grows proportion to its current population, the initial population is 5000 and grows 4%per year this can be modeled by $\frac{dN}{dt} = 0.04N$, $N(0) = 5000$

(a) Find an equation to model the population.
(b) Determine the population after 3 years.
(c) Determine how long will take the population to be double.

Solution:

(a) We are asked to find an explicit formula for N in terms of t ,but the differential equation is already given by $\frac{dN}{dt} = 0.04N$

Now solving the equation using separation of variables we have

$$\frac{dN}{N} = 0.04dt$$

Integrate both sides of the equation

$$\int \frac{1}{N} dN = \int 0.04dt + c$$

$$\ln N = 0.04t + c$$

$$N = e^{0.04t+c}$$

$$N(t) = N_0 e^{0.04t} \text{ where } e^c = N_0$$

Since the initial population of the solution was 5,000 and subsisting $t = 0$

$$N(0) = N_0 e^0 = 5{,}000 \text{ thus } N_0 = 5{,}000$$

So we have model of population

$$N(t) = 5{,}000e^{0.04t}$$

(b) Subsisting $t = 3$

$$N(t) = 5{,}000e^{0.04t}$$

$N(3) = 5{,}000e^{0.04(3)}$, the population after 3 years becomes

$$N(3) \approx 5637$$

(c) We wish to find out what is t when $N = 2(5000) = 10000$

Using the above equation

$N(t) = 5{,}000e^{0.04t}$ at $N(t) = 2(5000) = 10000$ we have

$$10000 = 5000e^{0.04t}$$

$$2 = e^{0.04t}$$

Taking $\ln$ both sides

$$\ln 2 = \ln e^{0.04t}$$

$$\ln 2 = 0.04t$$

$$t = 17.3 \text{ years}$$

Therefor after 17.3 years the population will be doubled to 1000.

❖ If immigration and emigration are including, equation (12) takes the form of a linear first ordinary differential equation,

$$\left.\begin{aligned} N'(t) &= \rho(t)N + I(t), \\ \rho(t) &= \beta(t) - \mu(t) - \eta(t) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The differential equation in equation (17) is called a first order non homogeneous linear differential equation for open population.

The equations in (17) can be solved by using an integrating factor. We collect the N terms on one side of the equation and multiply by some function $F(t)$ (the integrating factor);

$$F(t)N'(t)-\rho(t)F(t)N(t)=F(t)I(t)$$

If $F'(t)=-\rho(t)F(t)$, Then

$$F(t)N'(t)-F'(t)N(t)=F(t)I(t)$$

Then the product rule allows us to write this equation as;

$$\frac{d}{dt}(F(t)N(t))=F(t)I(t)$$

The first equation is the form (14), so a special solution is;

$$F(t)=\exp\left(-\int_0^t \rho(s)ds\right)=\frac{1}{Q(t)},$$

Integrating the second equation gives

$$F(t)N(t)-F(t_0)N(t_0)=\int_{t_0}^{t} F(t)I(s)ds,$$

We solve for N i.e.

$$F(t)N(t)=F(t_0)N(t_0)+\int_{t_0}^{t} F(t)I(s)ds,$$

$$N(t)=\frac{F(t_0)N(t_0)}{F(t)}+\int_{t_0}^{t}\frac{F(s)I(s)}{F(t)}ds$$, thus the general solution is

$$\left.\begin{aligned} N(t)&=N_0\frac{Q(t)}{Q(t_0)}+\int_{t_0}^{t} I(s)\frac{Q(t)}{Q(s)}ds,\\ Q(t)&=\exp\left(\int_0^t \rho(s)ds\right)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(**)$$

Hence (**) is the general solutions of equation (17). Again, in the formula for the fundamental solution Q, the lower integration limit 0 can be replaced by affixed, but arbitrary, number r_0 which can be chosen at convenience. In other words, we can choose Q as the exponential of any anti-derivative of ρ.

4. Conclusion

Generally, this paper gives the following conclusion

- ✓ Constructing population growth model for single species (open and closed population) in differential equation form and solving it using separable variable method.
- ✓ How populations grow when they have un limited resource, but they will ultimately be limited by resource.
- ✓ Exponential growth takes place when a population's per capita growth rate stays the same regardless of population size, making the population grow faster and faster as it gets larger.
- ✓ We have seen that the application of first order differential equation on population dynamics to calculate the growth and decay problems and the most appropriate method in solving first order differential equation for closed population is variable separable.

References

[1]. Anderson, R.M., 1982b, transmission dynamics and control of infectious disease agents, *population biology of infectious diseases* (ed. R.M. Anderson, R. M. May),pp.1-12.life sciences Research Report,no.25,Springer, Berlin.

[2]. Bankcard, Paul, Robbery L. Devaney and Glen r. Hall. *Differential equation brooks* Cole publishing: Washington, 1998.

[3]. Bird J. O. - High Engineering Mathematics. Third Edition. Great Britain by Martins the Printers Ltd, India. 532-542, 2002

[4]. Cooke, K.L. 1967, functional differential equations: some models and perturbation problems, *differential equations and dynamical systems* (ed. J. K. Hale, J. P. LaSalle), pp. 167-183, Academic press, New York.

[5]. Edelstein-Keshet, L., 1988, *Mathematical models in biology*, McGraw-Hill, New York.

[6]. Gabriel B.Costa and Richard Bronson: *Differential Equations*; 3rd edition , McGraw-Hill, New York, 2006

[7]. Heathcote, H.W., and Van Ark, J.W., 1992, *Modeling HIV transmission and AIDS in the United States*, Lecture notes in biomathematics, vol. 75,205-227.

[8]. Michael et al.- Bay Delta forum in Importance of Water Temperature in Aquatic Systems. Delta, 1995

[9]. Preston, S.H., Heuveline, P., and Guillot, M., 2001, Demography. Measuring and modeling population processes, Blackwell

[10]. Sasser J. E.- History of Ordinary Differential Equation First Hundred Years, 2005.

[11]. Stanley I. G. - Elementary Differential with Application, Addison Wesley Publishing Company, Canada. 231-234, 1981.

[12]. W.E. Boyce &R.C. Diprima. *Elementary Differential Equations and boundary value problems*, 7th edition. John Wiley & Sons, INC., 2001