# OBSERVATIONS ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $9x^2+2y^2=27z$

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***Abstract :*** The ternary quadratic equation given by $9x^2+2y^2=27z$ is considered and searched for its many different integer solutions.Five different choices of integer solutions of the above equation are presented. A few interesting relations between the solutions and special polygonal numbers are presented.
**key words:** ternary quadratic, integer solution
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## I. INTRODUCTION:

The Diophantine equation offer an unlimited field for research due to their variety [1-3] in particular one may refer[4-14] for quadratic equations with three unknowns. This communication concerns with yet another interesting equations $9x^2+2y^2=27z$ for determining many non-zero integral points. Also, a few interesting relations among the solutions are presented.

## II. NOTATIONS:

1. $t_{m,n}$ =n[th] term of a regular polygon with m sides
$$=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$
2. $S_n=6n(n-1)+1$ = Star number of rank n
3. $pr_n=n(n+1)$ = Pronic number of rank n

## III. METHOD OF ANALYSIS:

THE TERNARY QUADRATIC DIOPHANTINE EQUATION TO BE SOLVED FOR ITS NON-ZERO DISTINCT INTEGRAL SOLUTION IS

$$9x^2+2y^2=27z \quad (1)$$

Substituting

$$y=3Y \quad (2)$$

in (1), we get

$$x^2+2Y^2=3z \quad (3)$$

(3) is solved through different approaches and the different patterns of solution (1) obtained are presented below

**PATTERN : 1**

Assume

$$z=\left(a^2+2b^2\right)^2 \quad (4)$$

Write 3 as,

$$3=\left(1+i\sqrt{2}\right)\left(1-i\sqrt{2}\right) \quad (5)$$

Using (4) and (5) in (3) we get

$$\left(x+i\sqrt{2}Y\right)\left(x-i\sqrt{2}Y\right)=\left(1+i\sqrt{2}\right)\left(1-i\sqrt{2}\right)\left(a+i\sqrt{2}b\right)^2\left(a-i\sqrt{2}b\right)^2$$

Consider the positive factor,

$$\left(x+i\sqrt{2}Y\right)=\left(1+i\sqrt{2}\right)\left(a+i\sqrt{2}b\right)^2$$

$$\left(x+i\sqrt{2}Y\right)=a^2-2b^2+i2\sqrt{2}ab+i\sqrt{2}a^2-i2\sqrt{2}b^2-4ab$$

$$\left(x+i\sqrt{2}Y\right)=\left(a^2-2b^2-4ab\right)+i\sqrt{2}\left(2ab+a^2-2b^2\right)$$

Equating real and imaginary parts,

$$x=a^2-2b^2-4ab$$

$$Y = a^2 - 2b^2 + 2ab$$

Using (2)

$$y = 3a^2 - 6b^2 + 6ab$$

We obtain the non- zero distinct integral solution of (1) as

$$x(a,b) = a^2 - 2b^2 - 4ab$$

$$y(a,b) = 3a^2 - 6b^2 + 6ab$$

$$z(a,b) = (a^2 + 2b^2)^2$$

**PROPERTIES:**

1. $y(1,b) + t_{14,b} - \Pr_b + t_{4,b} - 3 = 0$
2. $y(1,b) + t_{14,b} + t_{6,b} - 2t_{4,b} \equiv 0 \pmod 3$
3. $y(1,b) + t_{14,b} - G_b + \Pr_b - t_{4,b} - 4 = 0$
4. $y(1,a) - S_a + t_{8,a} - 10\Pr_a + 10t_{4,a} \equiv 0 \pmod 7$
5. $x(a,1) - t_{a,4} + 2G_a + 4 = 0$

**PATTERN : 2**

3 can also be written as

$$3 = \frac{(5 + i\sqrt{2})(5 - i\sqrt{2})}{3^2} \qquad (5)$$

Substituting (5) in (4) and employing the method of factorization We get

$$(x + i\sqrt{2}Y)(x - i\sqrt{2}Y) = (a + i\sqrt{2}b)^2 (a - i\sqrt{2}b)^2 * \frac{(5 + i\sqrt{2})(5 - i\sqrt{2})}{3^2}$$

Consider the positive factor

$$(x + i\sqrt{2}Y) = (a + i\sqrt{2}b)^2 * \frac{(5 + i\sqrt{2})}{3}$$

$$(x + i\sqrt{2}Y) = (a^2 - 2b^2 + i\sqrt{2}ab) * \frac{(5 + i\sqrt{2})}{3}$$

$$(x + i\sqrt{2}Y) = \frac{(5a^2 - 10b^2 - 4ab) + i\sqrt{2}(a^2 - 2b^2 + 10ab)}{3}$$

Equating real and imaginary parts of the above equation we get

$$x(a,b) = \frac{5a^2 - 10b^2 - 4ab}{3}$$

$$Y(a,b) = \frac{a^2 - 2b^2 + 10ab}{3}$$

Using (2)

$$y(a,b) = \frac{3a^2 - 6b^2 + 30ab}{3}$$

$$z(a,b)=\left(a^2+2b^2\right)^2$$

Assume a=3A and b=3B in the above equation the non – zero distinct integral solution of (1) is given by

$$x(A,B)=15A^2-30B^2-12AB$$

$$y(A,B)=9A^2-18B^2+90AB$$

$$z(A,B)=\left(9A^2+18B^2\right)^2$$

**PROPERTIES:**

1. $x(A,1)-t_{15,A}-2\Pr_A+2t_{4,A}\equiv 0(\bmod 5)$

2. $x(1,B)+t_{62,B}+41\Pr_B-41t_{4,B}-15=0$

3. $y(A,1)-t_{20,A}-98\Pr_A+98t_{4,A}\equiv 0(\bmod 2)$

4. $y(1,B)+t_{38,B}-73\Pr_B+73t_{4,B}-9=0$

5. $x(1,B)+y(1,B)+t_{98,B}+125\Pr_B-125t_{4,B}-24=0$

**PATTERN : 3**

(4) can also be written as

$$x^2+2Y^2=3z*1 \qquad (6)$$

Write 1 as

$$1=\frac{\left(1+i2\sqrt{2}\right)\left(1-i2\sqrt{2}\right)}{3^2} \qquad (7)$$

Substituting (7) in (6) and employing the method of factorization we get,

$$\left(x+i\sqrt{2}Y\right)\left(x-i\sqrt{2}Y\right)=\left(1+i\sqrt{2}\right)\left(1-i\sqrt{2}\right)\left(a+i\sqrt{2}b\right)^2\left(a-i\sqrt{2}b\right)^2*\frac{\left(1+i2\sqrt{2}\right)\left(1-i2\sqrt{2}\right)}{3^2}$$

Consider the positive factor,

$$\left(x+i\sqrt{2}Y\right)=\left(1+i\sqrt{2}\right)\left(a+i\sqrt{2}b\right)^2*\frac{\left(1+i2\sqrt{2}\right)}{3}$$

$$\left(x+i\sqrt{2}Y\right)=\left(1+i\sqrt{2}\right)\left(a^2-b^2+i2ab\sqrt{2}\right)*\frac{\left(1+i2\sqrt{2}\right)}{3}$$

$$\left(x+i\sqrt{2}Y\right)=\frac{\left(-3a^2+6b^2-12ab\right)+i\sqrt{2}\left(3a^2-6b^2-6ab\right)}{3}$$

Equating real and imaginary parts,

$$x = \frac{(-3a^2 + 6b^2 - 12ab)}{3}$$

$$Y = \frac{(3a^2 - 6b^2 - 6ab)}{3}$$

Using (2)

$$y = \frac{(9a^2 - 18b^2 - 18ab)}{3}$$

We obtain the non-zero distinct integral solution of (1) as

$$x(a,b) = (-a^2 + 2b^2 - 4ab)$$

$$y(a,b) = (3a^2 - 6b^2 - 6ab)$$

$$z(a,b) = (a^2 + 2b^2)^2$$

**PROPERTIES :**

1. $x(1,b) - t_{6,b} + 3\Pr_b - 3t_{4,b} + 1 = 0$

2. $x(a,1) + t_{4,a} + 4\Pr_a - 4t_{4,a} - 2 = 0$

3. $y(1,b) + S_b + 12\Pr_b - 12t_{4,b} \equiv 0 \pmod{2}$

4. $y(a,1) - t_{8,a} + 4\Pr_a - 4t_{4,a} \equiv 0 \pmod{3}$

5. $x(a,1) + y(a,1) - t_{6,a} + 9\Pr_a - 9t_{4,a} + 4 = 0$

**PATTERN : 4**

Substituting the linear transformation

$$\left.\begin{array}{l} x = X + 2T \\ Y = X - T \\ z = 6u^2 \end{array}\right\} \quad (8) \text{ in } (1)$$

we get,

$$(X + 2T)^2 + 2(X - T)^2 = 3 * 6u^2$$

$$X^2 + 4T^2 + 4XT + 2X^2 + 2T^2 - 4XT = 18u^2$$

$$3X^2 + 6T^2 = 18u^2$$

Which can also be written as

$$(X + 2u)(X - 2u) = 2(u + T)(u - T) \quad (9)$$

**CASE:1**

(9) can be written in the form of ratio as

$$\frac{X+2u}{2(u-T)}=\frac{u+T}{X-2u}=\frac{\alpha}{\beta} \tag{10}$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned}\beta X+2\alpha T+(2\beta-2\alpha)u=0\\ -\alpha X+\beta T+(\beta+2\alpha)u=0\end{aligned}\right\} \tag{11}$$

Solving (11) by method of cross multiplication we get

$$\left.\begin{aligned}X&=4\alpha^2-2\beta^2+4\alpha\beta\\ T&=2\alpha^2-\beta^2-4\alpha\beta\\ u&=2\alpha^2+\beta^2\end{aligned}\right\} \tag{12}$$

Substituting (12) in (8) the non –zero distinct integer solutions of (1) are given by,

$$x(\alpha,\beta)=8\alpha^2-4\beta^2-4\alpha\beta$$

$$y(\alpha,\beta)=6\alpha^2-3\beta^2+24\alpha\beta$$

$$z(\alpha,\beta)=6(2\alpha^2+\beta^2)^2$$

**PROPERTIES :**

1. $x(\alpha,1)-t_{18,\alpha}-3\Pr_{\alpha}+3t_{4,\alpha}\equiv 0(\bmod 4)$

2. $x(1,\beta)+t_{10,\beta}+7\Pr_{\beta}-7t_{4,\beta}-8=0$

3. $y(1,\beta)+t_{8,\beta}-22\Pr_{\beta}+22t_{4,\beta}\equiv 0(\bmod 3)$

4. $y(\alpha,1)-t_{14,\alpha}-29\Pr_{\alpha}+29t_{4,\alpha}+3=0$

5. $x(\alpha,1)+y(\alpha,1)-t_{30,\alpha}-33\Pr_{\alpha}+33t_{4,\alpha}+7=0$

**CASE:2**

(9) can also be written in the form of ratio as

$$\frac{X+2u}{u+T}=\frac{2(u-T)}{X-2u}=\frac{\alpha}{\beta}$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned}\beta X-\alpha T+(2\beta-\alpha)u=0\\ -\alpha X-2\beta T+(2\beta+2\alpha)u=0\end{aligned}\right\} \tag{13}$$

Solving (13) by method of cross multiplication we get,

$$\left.\begin{aligned}X&=-2\alpha^2+4\beta^2-4\alpha\beta\\ T&=\alpha^2-2\beta^2-4\alpha\beta\\ u&=-2\beta^2-\alpha^2\end{aligned}\right\} \tag{14}$$

Substituting (14) in (8) the non zero distinct integer solutions of (1) are given by,

$x(\alpha,\beta) = -12\alpha\beta$

$y(\alpha,\beta) = -9\alpha^2 + 18\beta^2$

$z(\alpha,\beta) = 6(-\alpha^2 - 2\beta^2)^2$

**PROPERTIES :**

1.$x(1,\beta) + 12\Pr_\beta - 12t_{4,\beta} = 0$

2.$y(1,\beta) - 6t_{4,\beta} + 3 = 0$

3.$x(1,\beta) + y(1,\beta) - t_{14,\beta} + 7\Pr_\beta - 7t_{4,\beta} \equiv 0 \pmod 3$

4.$y(\alpha,1) + 3t_{4,\beta} \equiv 0 \pmod 2$

5.$x(1,\beta) - y(1,\beta) - 6t_{4,\beta} + 12\Pr_\beta - 3 = 0$

**CASE:3**

Write (9) in the form of ratio as

$$\frac{X-2u}{2(u-T)} = \frac{u+T}{X+2u} = \frac{\alpha}{\beta}$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned}\beta X + 2\alpha T - (2\beta + 2\alpha)u = 0\\ -\alpha X + \beta T + (\beta - 2\alpha)u = 0\end{aligned}\right\} \qquad (15)$$

Solving (15) by method of cross multiplication we get

$$\left.\begin{aligned}X &= -4\alpha^2 + 2\beta^2 + 4\alpha\beta\\ T &= 2\alpha^2 - \beta^2 + 4\alpha\beta\\ u &= 2\alpha^2 + \beta^2\end{aligned}\right\} \qquad (16)$$

Substituting (16) in (8) the non zero distinct integer solution of (1) are given by

$x(\alpha,\beta) = 12\alpha\beta$

$y(\alpha,\beta) = -18\alpha^2 + 9\beta^2$

$z(\alpha,\beta) = 6(2\alpha^2 + \beta^2)^2$

**PROPERTIES :**

1.$x(1,\beta) - 12\Pr_\beta + 12t_{4,\beta} = 0$

2.$y(1,\beta) - 9t_{4,\beta} + 18 = 0$

3.$y(\alpha,1) + 18t_{4,\alpha} - 9 = 0$

4.$x(\alpha,1) + y(\alpha,1) + t_{38,\alpha} + 5\Pr_\alpha - 5t_{4,\alpha} \equiv 0 \pmod 9$

5.$x(\alpha,1) - y(\alpha,1) - t_{38,\alpha} - 29\Pr_\alpha + 29t_{4,\alpha} \equiv 0 \pmod 3$

**PATTERN : 5**

Substituting the linear transformation,

$$\left.\begin{aligned}x &= X + 2T\\ Y &= X - T\\ z &= 11u^2\end{aligned}\right\} \qquad (17)$$

in (1) we get,

$$(X + 2T)^2 + 2(X - T)^2 = 3*11u^2$$

$$X^2+4T^2+4XT+2X^2+2T^2-4XT=33u^2$$

$$3X^2+6T^2=33u^2$$

which can also be written as

$$(X+3u)(X-3u)=2(u+T)(u-T) \tag{18}$$

**CASE:1**

(18) can be written in the form of ratio as

$$\frac{(X+3u)}{2(u+T)}=\frac{(u-T)}{(X-3u)}=\frac{\alpha}{\beta} \tag{19}$$

which is equivalent to the system of double equations

$$\left.\begin{array}{l}\beta X-2\alpha T+(3\beta-2\alpha)u=0\\ -\alpha X-\beta T+(\beta+3\alpha)u=0\end{array}\right\} \tag{20}$$

Solving (20) by method of cross multiplication we get,

$$\left.\begin{array}{l}X=-6\alpha^2+3\beta^2-4\alpha\beta\\ T=2\alpha^2-\beta^2-6\alpha\beta\\ u=-2\alpha^2-\beta^2\end{array}\right\} \tag{21}$$

Substituting (21) in (17) the non-zero distinct integer solution of (1) are given by

$$x(\alpha,\beta)=-2\alpha^2+\beta^2-16\alpha\beta$$

$$y(\alpha,\beta)=-24\alpha^2+12\beta^2+6\alpha\beta$$

$$z(\alpha,\beta)=11(-2\alpha^2-\beta^2)^2$$

**PROPERTIES :**

1. $x(\alpha,1)+t_{6,\alpha}+17\Pr_\alpha-17t_{4,\alpha}-1=0$

2. $x(1,\beta)-t_{4,\beta}+16\Pr_\beta-16t_{4,\beta}+2=0$

3. $y(\alpha,1)+t_{50,\alpha}+17\Pr_\alpha-17t_{4,\alpha}\equiv 0 \pmod 6$

4. $y(1,\beta)-t_{26,\beta}-17\Pr_\beta+17t_{4,\beta}+24=0$

5. $x(1,\beta)+y(1,\beta)-t_{28,\beta}-2\Pr_\beta+2t_{4,\beta}+26=0$

**CASE:2**

(18) can be written in the form of ratio as

$$\frac{X+3u}{u-T}=\frac{2(u+T)}{X-3u}=\frac{\alpha}{\beta} \tag{22}$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned} &\beta X-\alpha T+(2\beta-\alpha)u=0\\ &-\alpha X-2\beta T+(2\beta+2\alpha)=0 \end{aligned}\right\} \tag{23}$$

Solving (23) by method of cross multiplication we get

$$\left.\begin{aligned} &X=3\alpha^2-6\beta^2+4\alpha\beta\\ &T=\alpha^2-2\beta^2-6\alpha\beta\\ &u=2\beta^2+\alpha^2 \end{aligned}\right\} \tag{24}$$

Substituting (24) in (17) the non zero distinct integer solution of (1) are given by

$$x(\alpha,\beta)=5\alpha^2-10\beta^2-8\alpha\beta$$

$$y(\alpha,\beta)=6\alpha^2-12\beta^2+30\alpha\beta$$

$$z(\alpha,\beta)=11(\alpha^2+2\beta^2)^2$$

**PROPERTIES:**

1. $x(1,\beta)+t_{22,\beta}+17\,\mathrm{Pr}_\beta-17t_{4,\beta}-5=0$

2. $x(\alpha,1)-t_{12,\alpha}+4\,\mathrm{Pr}_\alpha-4t_{4,\alpha}+10=0$

3. $y(\alpha,1)-t_{14,\alpha}-35\,\mathrm{Pr}_\alpha+35t_{4,\alpha}+12=0$

4. $y(1,\beta)+t_{26,\beta}-19\,\mathrm{Pr}_\beta+19t_{4,\beta}\equiv 0 \pmod 6$

5. $x(\alpha,1)+y(\alpha,1)-t_{24,\alpha}-32\,\mathrm{Pr}_\alpha+32t_{4,\alpha}\equiv 0 \pmod{22}$

**CASE:3**

(18) can be written in the form of ratio as

$$\frac{X-3u}{u-T}=\frac{2(u+T)}{X+3u}=\frac{\alpha}{\beta} \tag{25}$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned} &\beta X+\alpha T-(3\beta+\alpha)u=0\\ &-\alpha X+2\beta T+(2\beta-3\alpha)u=0 \end{aligned}\right\} \tag{26}$$

Solving (26) by method of cross multiplication we get

$$\left.\begin{array}{l} X = -3\alpha^2 + 6\beta^2 + 4\alpha\beta \\ T = \alpha^2 - 2\beta^2 + 6\alpha\beta \\ u = \alpha^2 + 2\beta^2 \end{array}\right\} \quad (27)$$

Substituting (27) in (17) the non zero distinct integer solution of (1)

are given by

$$x(\alpha,\beta) = -\alpha^2 + 2\beta^2 + 16\alpha\beta$$

$$y(\alpha,\beta) = -12\alpha^2 + 24\beta^2 - 6\alpha\beta$$

$$z(\alpha,\beta) = 11(\alpha^2 + 2\beta^2)$$

**PROPERTIES:**

1. $x(\alpha,1) + t_{4,\alpha} - 16\,\Pr_\alpha + 16t_{4,\alpha} - 2 = 0$

2. $x(1,\beta) - t_{6,\beta} - 17\,\Pr_\beta + 17t_{4,\beta} + 1 = 0$

3. $y(1,\beta) - t_{50,\beta} - 17\,\Pr_\beta + 17t_{14,\beta} + 12 = 0$

4. $y(\alpha,1) + t_{26,\alpha} + 17\,\Pr_\alpha - 17t_{4,\alpha} \equiv 0 \pmod{12}$

5. $x(\alpha,1) + y(\alpha,1) + t_{28,\alpha} + 2\,\Pr_\alpha - 2t_{4,\alpha} \equiv 0 \pmod{13}$

**iv.Conclusion:**

In this paper we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by $9x^2 + 2y^2 = 27z$ . As quadratic equations are rich in variety one may search for other choices quadratic equation with variables greater than or equal three and determine their properties through special numbers

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