



Bulk Viscous Effects on Logamediate Expansion within the $f(R)$ Gravity

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Abstract:

In this work, we investigate late-time cosmic dynamics within the framework of $f(R)$ gravity by considering a spatially flat Friedmann–Robertson–Walker (FRW) universe filled with a bulk viscous cosmic fluid. A logarithmic scale factor of the form $a(t) = \exp[\beta(\ln t)^\alpha]$, with $\alpha > 1$ and $\beta > 0$, was employed to model the accelerated expansion of the universe. The modified Friedmann equations were derived using the metric formalism of $f(R)$ gravity, and exact expressions for key cosmological quantities such as the Hubble parameter, Ricci scalar, and effective energy density and pressure were obtained. The effects of bulk viscosity on cosmic evolution were analysed through different viscosity prescriptions. The dynamical behaviour of the model is further characterized using the deceleration parameter and state-finder diagnostics, which demonstrate a smooth transition to an accelerated phase and convergence toward the Λ CDM fixed point at later times.

Keywords: $f(R)$ Gravity, Logarithm scale factor, Bulk viscous fluid, LFRW metric, Sate-finder parameter.

1. Introduction:

The late-time accelerated expansion of the universe, first indicated by Type Ia supernovae observations [1–3] and later supported by CMB anisotropies and baryon acoustic oscillations [4,5], has motivated extensive investigations into gravitational modifications and exotic matter sources beyond the standard Λ CDM paradigm. Among the many modified gravity theories, $f(R)$ gravity has emerged as a compelling generalization of General Relativity (GR), allowing the Ricci scalar R in the Einstein–Hilbert action to be replaced by an arbitrary function $f(R)$ [6–9]. This modification not only provides a geometric description of cosmic acceleration without explicitly invoking dark energy but also offers rich phenomenology in both early- and late-time cosmology.

On the matter side, bulk viscous cosmological fluids have gained considerable attention as an effective mechanism capable of generating negative pressure through dissipative processes [10–13]. The presence of bulk viscosity naturally arises in the context of relativistic fluid dynamics, kinetic theory, and nonequilibrium thermodynamics, and has been used to successfully describe inflation, reheating, and late-time acceleration

within various gravitational frameworks [14–17]. In an expanding FRW universe, the bulk viscous pressure modifies the effective equation of state, leading to accelerated expansion, even in the absence of exotic fields. In parallel, the logarithmic expansion introduced by Barrow [18] represents an intermediate form of cosmic evolution that lies between power-law and exponential inflation. Scale factor of the form

$$a(t) = \exp[\beta(\ln t)^\alpha], \alpha > 1, \beta > 0, \quad (1)$$

leads to accelerated expansion without requiring extreme fine-tuning, and accommodates a wide variety of inflationary and late-time behaviours. This scale factor has been widely used to analyse inflationary observables, dark energy dynamics, holographic cosmology, and anisotropic models [19–22].

Combining these three ingredients, $f(R)$ gravity, bulk viscous fluid, and the logarithmic scale factor within an FRW background, offers a powerful framework for modeling the full cosmic evolution, from early inflation to the current acceleration. The interplay between geometric modifications through $f(R)$, dissipative effects through bulk viscosity, and non-power-law expansion through logamediate evolution can potentially produce richer dynamics than any single component alone. Recent studies have shown that dissipative modified gravity models can satisfy energy conditions, exhibit stable behaviour, and remain compatible with the current observational bounds [23–27]. However, a comprehensive investigation of the logamediate evolution in $f(R)$ cosmology with bulk viscous fluid remains unexplored. Motivated by these considerations, the present study aims to investigate the cosmological dynamics of an FRW universe with bulk viscous matter within $f(R)$ gravity, assuming a logarithmic scale factor. We analyze the behavior of key physical quantities, including the Hubble parameter, energy density, pressure, and effective equation of state, and discuss their implications in the context of the latest cosmic acceleration scenario. This model provides insights into the role of dissipative processes in modified gravity and sheds light on the viability of logamediate expansion as an alternative to standard inflationary and dark-energy models.

2. A Brief Review of the $f(R)$ Gravity:

The $f(R)$ theory of gravity represents a natural and well-motivated extension of General Relativity (GR), obtained by generalizing the Einstein–Hilbert action by replacing the Ricci scalar R with an arbitrary function $f(R)$ [6,7,9]. The modified action is expressed as follows:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m, \quad (2)$$

where $S_m = \int d^4x L_m g_{\mu\nu}$ denotes the matter action and $\kappa^2 = 8\pi G$. This modification introduces higher-order curvature contributions to the gravitational sector, allowing for richer cosmological dynamics while retaining the geometric foundation of the GR.

The variation of the above action with respect to the metric yields fourth-order field equations

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = \kappa^2 T_{\mu\nu}, \quad (3)$$

where $F(R) = \frac{df(R)}{dR}$, which reduces to Einstein's field equations when $f(R) = R + \lambda R^n$. The presence of the derivative terms $\nabla_\mu \nabla_\nu F$ and $\square F$ highlights the higher-derivative nature of the theory and introduces an effective scalar degree of freedom, often described through a conformal transformation to the Einstein frame [8,23,28].

3. The Cosmological FRW Model and Bulk Viscous fluid:

We work in a spatially flat Friedmann–Robertson–Walker (FRW) space–time with a line element [30] as follows:

$$ds^2 = -dt^2 + a^2(t)(dz^2), \quad (4)$$

where $a(t)$ is the scale factor and the Hubble parameter is $H(t) = \dot{a}/a$. The Ricci scalar for $k = 0$ is:

$$R = 6(\dot{H} + 2H^2). \quad (5)$$

We assume a bulk viscous fluid in relativistic cosmology, and a bulk viscous (dissipative) fluid is described by modifying the effective pressure using a viscous term. The energy–momentum tensor in Eckart (non-causal) theory is given by

$$T_{\mu\nu} = (\rho + \tilde{p})u_\mu u_\nu + \tilde{p} g_{\mu\nu}, \quad (6)$$

where ρ is the energy density of the fluid, p is the thermodynamic equilibrium pressure, $\tilde{p} = p + \Pi$ the effective pressure including viscous contribution, u^μ is the fluid velocity, and Π is the bulk viscous pressure (a negative quantity for cosmic expansion).

For a comoving observer in the FRW space–time, $u^\mu = (1, 0, 0, 0)$, $u_\mu u^\mu = -1$.

Under conditions of spatial homogeneity and isotropy (which refer to the cosmological principle, for instance, Reference [31]), the bulk viscous pressure is a unique admissible dissipative phenomenon. In a gas dynamical model, the existence of an effective bulk pressure can be traced back to a nonstandard self-interacting force on the particles of the gas [32]. The bulk viscosity contributes negatively to the total pressure for instance References [33–35].

Because matter is minimally coupled in metric $f(R)$ gravity, the usual conservation law holds.

$$\nabla^\mu T_{\mu\nu} = 0, \quad (7)$$

$$\dot{\rho} + 3H(\rho + \tilde{p}) = 0. \quad (8)$$

We adopt the Eckart ansatz for bulk viscosity $\Pi = -3\xi H$ then, $\tilde{p} = p - 3\xi H$ and the continuity equation becomes

$$\dot{\rho} + 3H(\rho + p - 3\xi H) = 0, \quad \xi > 0. \quad (9)$$

4. Solution of the Field Equations:

To solve the field equation completely, using Eq. (1), we obtain the Hubble parameter as

$$H = \frac{\beta\alpha(\ln \ln t)^{\alpha-1}}{t}, \quad (10)$$

Implies,

$$\dot{H} = \frac{\beta\alpha}{t^2} [(\alpha - 1)(\ln t)^{\alpha-2} - (\ln t)^{\alpha-1}], \quad (11)$$

From Eq. (5), we have the following:

$$R = \frac{6\beta\alpha}{t^2} [(\alpha - 1)(\ln t)^{\alpha-2} - (\ln t)^{\alpha-1}] + \frac{12\beta^2\alpha^2(\ln t)^{2\alpha-2}}{t^2}, \quad (12)$$

For the above choice of the $f(R)$ function, we calculate the equation of density and pressure for the given model. From the modified Friedmann equations,

$$\rho_{eff} = \frac{1}{\kappa^2} \left[3FH^2 - \frac{1}{2}(FR - f) + 3H\dot{F} \right], \quad (13)$$

Initially, $\rho(t)$ increases rapidly and reaches a maximum, indicating a high-energy phase driven by the underlying modified gravity or viscous effects. After this peak, the energy density decreased smoothly as the universe expanded, reflecting the dilution of energy with time. The absence of any divergence indicates that the model avoids future singularities and remains dynamically stable. Later, the small but finite value of $\rho(t)$ is consistent with a dark-energy–dominated accelerating universe.

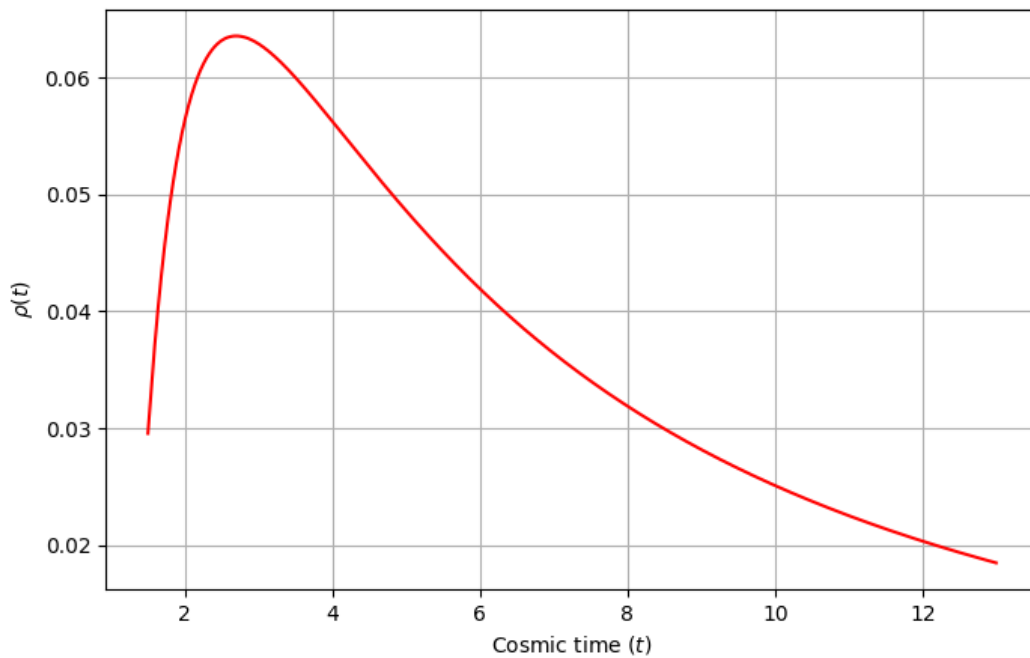


Figure 1: Energy density $\rho(t)$ v/s cosmic time (t) with $\alpha = 2.0$, $\beta = 1.0$, $\lambda = 0.01$.

Here, we have two types of pressures: effective pressure and isotropic pressure, as follows:

$$\tilde{p} = p - 3\xi H = \frac{1}{\kappa^2} [-2F\dot{H} - \ddot{F} + H\dot{F}] - \rho, \quad (14)$$

and

$$p = 3\xi H + \frac{1}{\kappa^2} [-2F\dot{H} - \ddot{F} + H\dot{F}] - \rho. \quad (15)$$

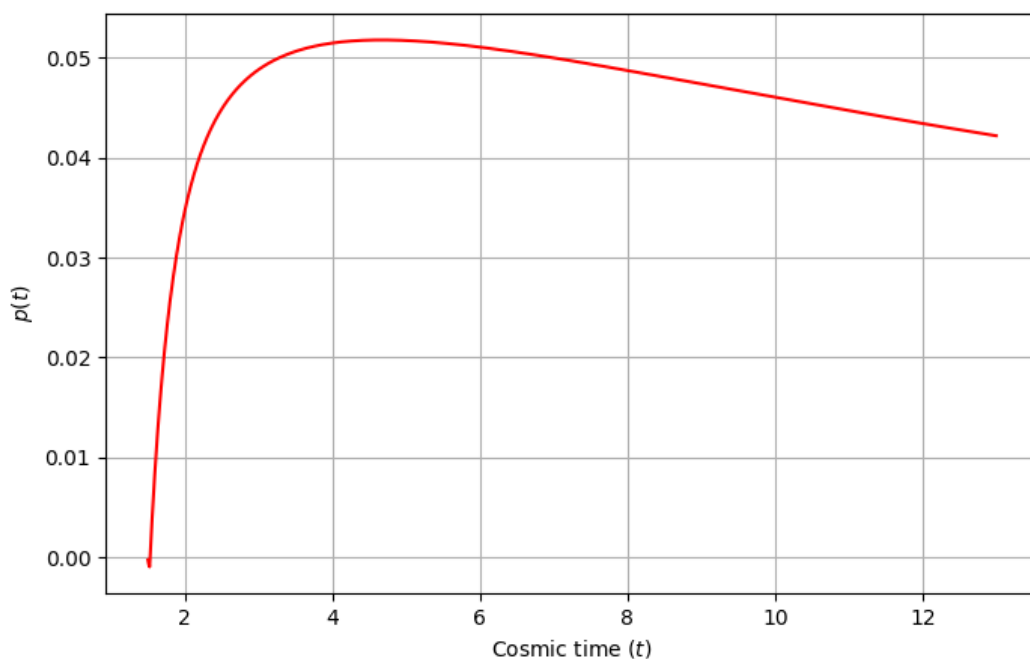


Figure 2: Isotropic pressure $p(t)$ v/s cosmic time (t) with $\alpha = 2.0$, $\beta = 1.0$, $\lambda = 0.01$, $\xi = 0.05$.

The pressure grows rapidly at early times, reflecting strong viscous and modified-gravity effects that contribute significantly to the cosmic fluid. It then reaches a mild maximum and decreases smoothly, indicating a gradual weakening of these effects as the universe expands. This behaviour is consistent with recent studies in viscous cosmology and $f(R)$ modified gravity models, where bulk viscosity yields an effective negative pressure capable of driving accelerated expansion. The smooth, non-singular evolution of $p(t)$ aligns with the current research favouring stable late-time acceleration without future singularities.

5. Physical and Kinematical properties:

The physical and kinematic behaviour of the considered $f(R)$ cosmological model is governed by the evolution of the Hubble parameter and its higher-order derivatives, which encode the expansion history of the universe. The kinematic and physical characteristics indicate that the $f(R)$ model offers a consistent and observationally viable description of the late-time cosmic acceleration. Considering the above motivation, we focus on the following two cases:

Case I: Constant bulk viscosity $\xi = \xi_0$

Eq. (9) becomes

$$\dot{\rho} + 3H(1 + \omega)\rho = 9\xi_0 H^2, \quad (16)$$

Substituting $H(t)$, we obtained

$$\dot{\rho} + 3(1 + \omega) \frac{\beta \alpha (\ln t)^{\alpha-1}}{t} \rho = \frac{9\xi_0 \beta^2 \alpha^2 (\ln t)^{2\alpha-2}}{t^2}, \quad (17)$$

This equation admits the solution

$$\rho(t) = \frac{3\xi_0 \beta \alpha (\ln t)^{\alpha-1}}{(1+\omega)t} + \rho_0 \exp[-3(1 + \omega)\beta (\ln t)^\alpha], \quad (18)$$

At late times, viscosity dominates and drives acceleration. The blue curve exhibits a rapid decay of ρ initially, followed by a smooth monotonic decrease. A constant viscosity contributes to a persistent negative effective pressure $p_{\square\square} = p - 3\xi_0 H$, which enhances the dissipation as the universe expands. This strong early time damping efficiently dilutes the energy density and drives the system quickly toward a low-density regime, consistent with accelerated expansion without late-time instabilities.

Case II: Hubble-dependent viscosity $\xi = \xi_0 H$

$$\xi = \xi_0 \frac{\beta \alpha (\ln t)^{\alpha-1}}{t}, \quad (19)$$

Eq. (9) becomes

$$\dot{\rho} + 3H(1 + \omega)\rho = 9\xi_0 H^3, \quad (20)$$

leading to

$$\rho(t) \propto \frac{(\ln \ln t)^{2\alpha-2}}{t^2}. \quad (21)$$

This model produces a strong late-time negative pressure. The red curve shows a milder early-time behaviour with a small peak before decay. When the viscosity scales with the expansion rate, viscous effects are weak at very early times, grow as H becomes significant, and then fade as H decreases. The transient rise in ρ reflects this delayed viscous contribution, after which dissipation dominates and the energy density decreases.

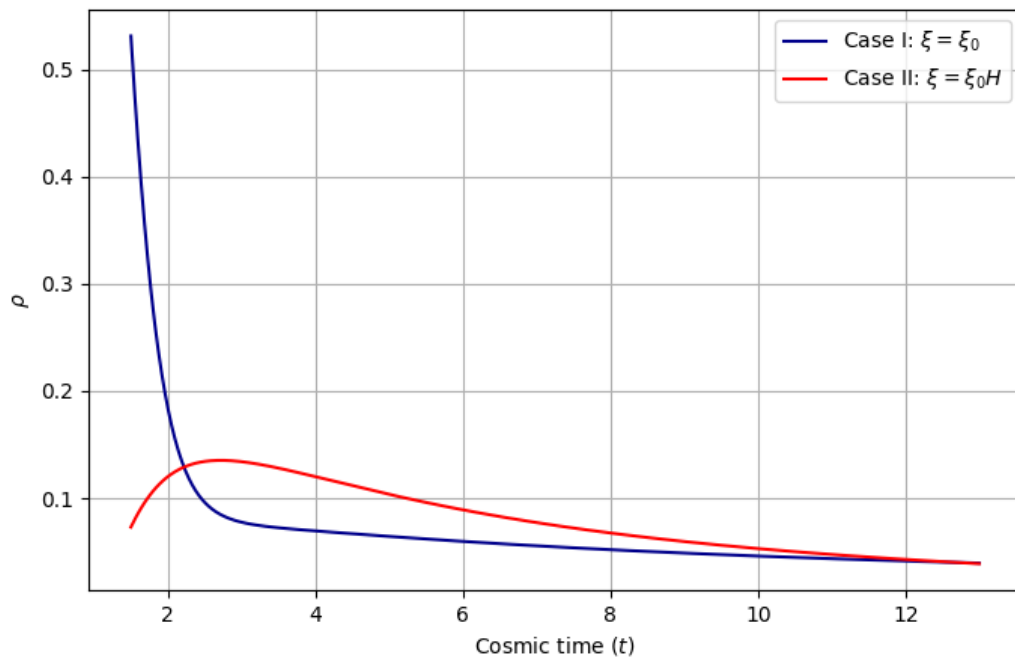


Figure 3: Different cases of $\rho(t)$ v/s cosmic time (t) with $\alpha = 2.0$, $\beta = 1.0$, $\omega = 0.5$, $\xi = 0.05$, $\rho_0 = 1.0$.

Later, both cases converge to a similar decay rate, indicating that viscous effects become subdominant and the universe approaches a smooth, stable expansion phase. The absence of divergences or oscillations suggests that both models are free from future singularities and are physically viable; the main difference lies in how bulk viscosity regulates the early to intermediate cosmic evolution.

Deceleration parameter q

$$q = -1 - \frac{\dot{H}}{H^2}, \quad (22)$$

Using Eqs. (10) – (11),

$$q(t) = -1 + \frac{1}{\beta\alpha(\ln t)^{\alpha-1}} - \frac{\alpha-1}{\beta\alpha(\ln t)^\alpha}. \quad (23)$$

At Late-time limit ($t \rightarrow \infty$), then $q \rightarrow -1$.

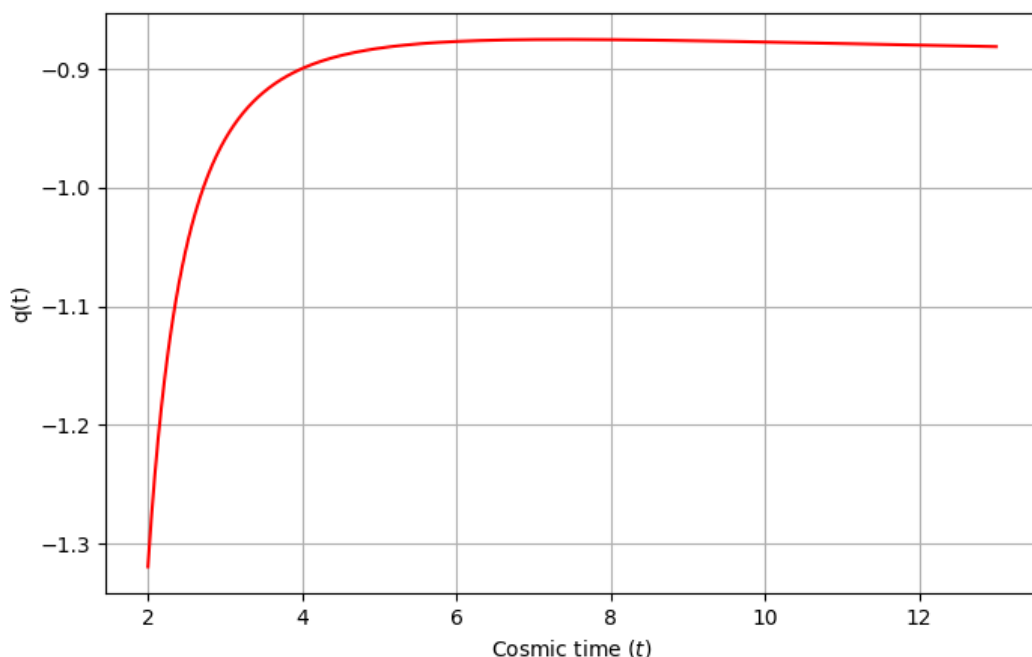


Figure 4: Deceleration parameter $q(t)$ v/s cosmic time (t) with $\alpha = 2.0$, $\beta = 1.0$.

Parameter remains negative throughout the evolution, confirming that the universe undergoes accelerated expansion at all times in this model. In the early stages, the strongly negative value of q indicated a phase of rapid acceleration driven by effective dark energy or modified gravity contributions. As the cosmic time increases, $q(t)$ approaches a nearly constant negative value, signalling a smooth transition to a stable late-time accelerating era. This behaviour is consistent with observationally favoured cosmological models, and supports the viability of the logarithmic expansion scenario.

State-finder parameter r and s

$$r = \frac{a'''}{aH^3} = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}, \quad \text{and} \quad s = \frac{r-1}{3(q-\frac{1}{2})}, \quad (24)$$

After substitution,

$$r(t) = 1 - \frac{3}{\beta\alpha(\ln t)^{\alpha-1}} + O((\ln t)^{-\alpha}), \quad (25)$$

The figure shows the evolution of jerk parameter $r(t)$, which characterizes departures from simple acceleration in the cosmic expansion. The rapid transition from negative to positive values followed by nearly constant behaviour indicates a smooth approach toward a stable accelerating phase, consistent with viable dark-energy or modified gravity models.

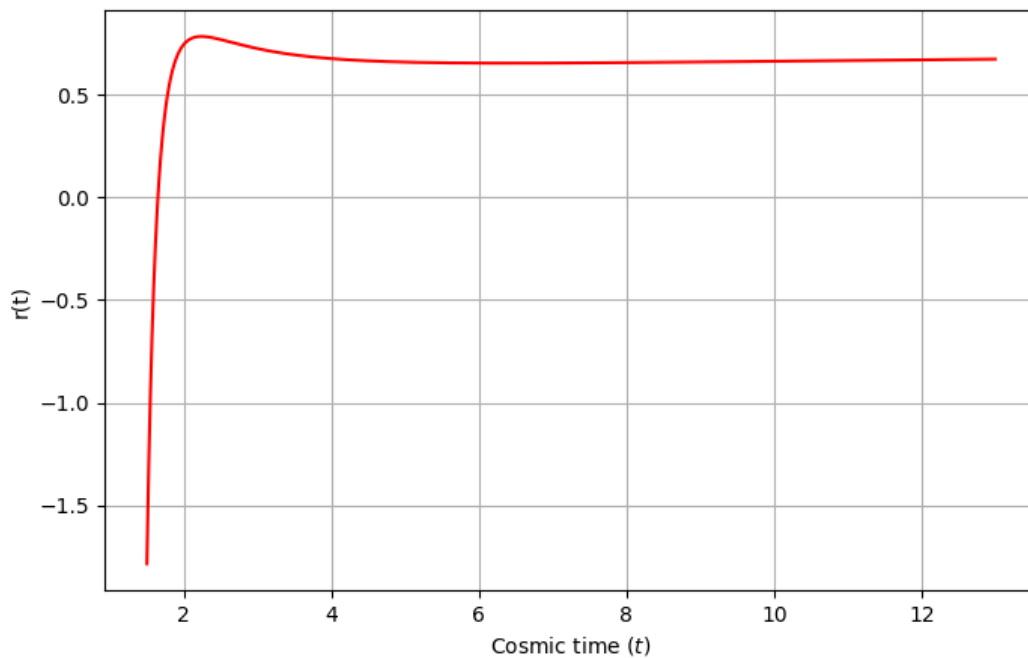


Figure 5: Jerk parameter $r(t)$ v/s cosmic time (t) with $\alpha = 2.0$, $\beta = 1.0$.

Substituting Eqs. (23) – (24) in Eq. (25),

$$s(t) = \frac{-\frac{3}{\beta\alpha(\ln t)^{\alpha-1}} + O((\ln t)^{-\alpha})}{3\left[-\frac{3}{2} + \frac{1}{\beta\alpha(\ln t)^{\alpha-1}} - \frac{\alpha-1}{\beta\alpha(\ln t)^{\alpha}}\right]}, \quad (26)$$

and to leading order at late times ($\ln t \gg 1$),

$$s(t) \simeq \frac{2}{\beta\alpha(\ln t)^{\alpha-1}} \rightarrow 0 \text{ as } (t \rightarrow \infty). \quad (27)$$

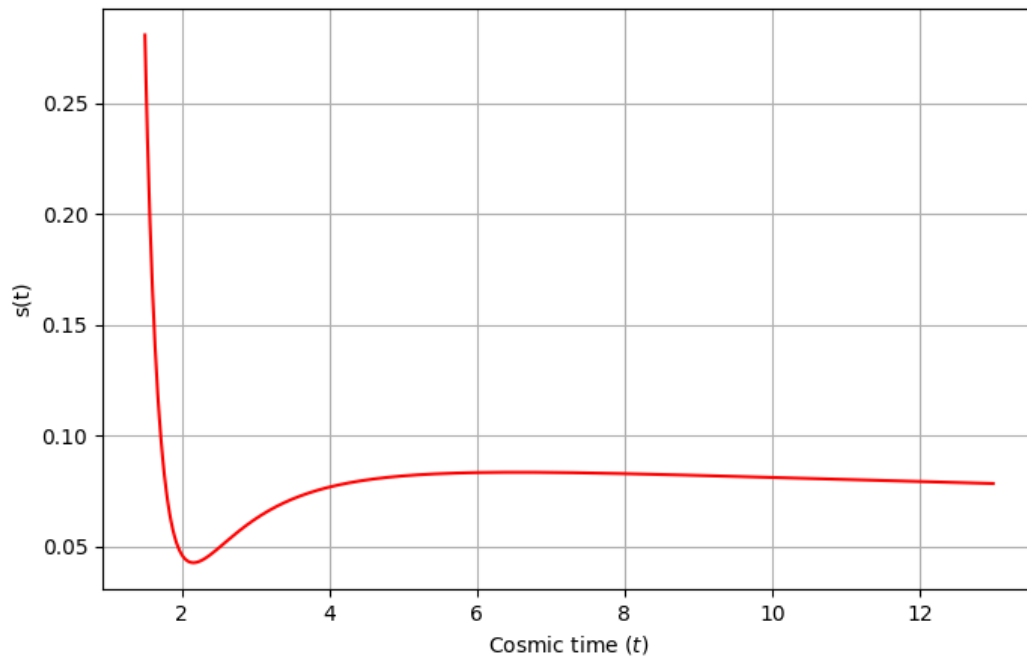


Figure 6: State-finder parameter $s(t)$ v/s cosmic time (t) with $\alpha = 2.0$, $\beta = 1.0$.

This shows that $s(t)$ decays to zero at late times, which is consistent with an approach toward a Λ CDM regime in the state-finder plane.

Late times: $(r, s) \rightarrow (1, 0)$. This indicates that the logarithmic viscous $f(R)$ cosmology mimics Λ CDM asymptotically.

The presence of bulk viscosity effectively alters the cosmic fluid pressure through a dissipative term proportional to the Hubble rate, generating an effective negative pressure that can drive late-time cosmic acceleration without introducing an explicit dark energy field. The logarithmic expansion law ensures smooth and continuous evolution of the Hubble parameter, enabling a gradual transition from decelerated to accelerated expansion. The kinematical behaviour, examined via state-finder diagnostics, shows convergence toward the Λ CDM fixed point $(r, s) = (1, 0)$ at late times, while allowing mild deviations at intermediate epochs.

Moreover, the interplay between the modified gravity effects and viscous dissipation suppresses the growth of curvature invariants, thereby preventing future finite-time singularities. Hence, viscous $f(R)$ gravity with logamediate expansion emerges as a stable and observationally consistent framework to explain the late-time acceleration of the Universe [36-42].

6. Conclusion:

The effective fluid formulation of the modified gravity model with bulk viscosity, energy density and pressure acquired geometric contributions through the functions $F(R)$ and their derivatives, whereas the viscous term introduces an effective negative pressure proportional to the Hubble parameter. For both Case I and Case II, the energy density $\rho(t)$ remains positive and evolves smoothly with cosmic time, dominated at later times by slowly varying logarithmic terms. The deceleration parameter $q(t)$ approaches -1 as $t \rightarrow \infty$, indicating a natural transition to an accelerated expansion phase without invoking an explicit dark energy component. The state-finder parameters further support this behaviour: the jerk parameter $r(t)$ tends to unity, whereas the parameter $s(t)$ decreases as $s(t) \simeq 2/[\beta\alpha(\ln t)^{\alpha-1}] \rightarrow 0$ at late times, signalling convergence toward the

Λ CDM fixed point model. Hence, the combined effects of logamediate expansion and bulk viscosity yield a dynamically consistent cosmological scenario that reproduces late-time acceleration and remains compatible with the standard observational diagnostics.

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